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# A Game-Theoretic Approach to Recommendation Systems with Strategic Content Providers

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## Abstract

We introduce a game-theoretic approach to the study of recommendation systems with strategic content providers. Such systems should be fair and stable. Showing that traditional approaches fail to satisfy these requirements, we propose the Shapley mediator. We show that the Shapley mediator satisfies the fairness and stability requirements, runs in linear time, and is the only economically efficient mechanism satisfying these properties.

## 1 Introduction

Recommendation systems (RSs hereinafter) have rapidly developed over the past decade. By predicting a user preference for an item, RSs have been successfully applied in a variety of applications. Moreover, the amazing RSs offered by giant e-tailers and e-marketing platforms such as Amazon and Google lie at the heart of online commerce and marketing in the web. However, current significant challenges faced by personal assistants (e.g. Cortana, Google Now and Alexa) and mobile applications go way beyond the practice of predicting the satisfaction levels of a user from a set of offered items. Such systems have to generate recommendations that satisfy the needs of both the end users and other parties or stakeholders [7, 40]. Consider the following cases:

- When Alice drives her car, her personal assistant runs the default navigation application. When she makes a stop in a junction, the personal assistant may show Alice advertisements provided by neighborhood stores, or an update on the stock market status as provided by financial brokers. Each of these information pieces — the plain navigation content, the local advertisements and the financial information — are served by different content providers. These content providers are all competing over Alice’s attention at a given point. The personal assistant is aware of Alice’s satisfaction with each content, and needs to select the right content to show at a particular time.
- Bob is reading news of the day on his mobile application. The application, aware of Bob’s interests, is presenting news deemed most relevant to these interests. The news is augmented by advertisements, provided by competing content providers, as well as articles by independent reporters. The mobile application, balancing Bob’s taste and the interests of the content providers, determines the mix of content shown to Bob.

In these contexts, the RS integrates information from various providers, often sponsored content, which is probably relevant to the user. The content providers are *strategic* — namely, make decisions based on the way the RS operates, aiming at maximizing their exposure. For instance, to draw Bob’s attention, a content provider strategically selects the topic of her news item, aiming at maximizing the

exposure to her item. On the one hand, fair content provider treatment is critical for smooth efficient use of the system and maintained content provider engagement over time. On the other hand, the strategic behavior of the content providers may lead to instability of the system: a content provider might choose to adjust the content she offers in order to increase the expected number of displays to the users, assuming the others stick to their offered contents.

In this paper we study ways of overcoming this dilemma using canonical concepts in game theory to impose two requirements on the RS: fairness and stability. Fairness is formalized as the requirement of satisfying fairness-related properties, and stability is defined as the existence of a pure Nash equilibrium. Analyzing RSs that satisfy these two requirements is the main goal of this paper.

Our first result is that traditional RSs fail to satisfy both of the above requirements. Traditional RSs are complete, in the sense that they always show some content to the user, but it turns out that this completeness property cannot be satisfied simultaneously with the fairness and equilibrium existence requirements. This impossibility result is striking and calls for a search for a fair and stable RS.

To do so, we model the setting as a cooperative game, binding content provider payoffs with user satisfaction. We resort to a core solution concept in cooperative game theory, the Shapley value [36], which is a celebrated mechanism for value distribution in game-theoretic contexts (see, e.g., [28]). In our work it is proposed as a tool for recommendations, namely for setting display probabilities. Since the Shapley value is employed in countless settings for fair allocation, it is not surprising that it satisfies our fairness properties. In addition, we prove that the related RS, termed the *Shapley mediator*, does satisfy the stability requirement. In particular, we show that the Shapley mediator possesses a potential function [27], and therefore any better-response learning dynamics converge to an equilibrium (see, e.g., [13, 17]). Note that this far exceeds our minimal stability requirement from the RS.

Implementation in commercial products would require the mediator to be computationally tractable. The mediator interacts with users, hence a fast response is of great importance. In another major result, we show that the Shapley mediator has a computationally efficient implementation. The latter is in contrast to the intractability of the Shapley value in classical game-theoretic contexts [14].

Another essential property of the Shapley mediator is economic efficiency [37]. Unlike cooperative games, where the Shapley value can be characterized as the only solution concept to satisfy properties equivalent to fairness and economic efficiency, in our setting the Shapley mediator is not characterized solely by fairness and economic efficiency. Namely, one can find other simple mediators that satisfy these two properties. However, we show that the Shapley mediator is the unique mediator to satisfy the fairness, economic efficiency and stability requirements.

Importantly, our study stems from a rigorous definition of the minimal requirements from an RS, and so characterizes a unique RS. Interested in understanding the ramification on user utility, we introduce a rigorous analysis of user utility in (strategic) recommendation systems, and show that the Shapley mediator is not-inferior to traditional approaches.

## 1.1 Related Work

This work contributes to three interacting topics: fairness in general machine learning, multi-stakeholder RSs and game theory.

The topic of fairness is receiving increasing attention in machine learning [5, 11, 30, 32] and data mining [23]. A major line of research is discrimination aware classification [16, 19, 21, 39], where classification algorithms must maintain high predictive accuracy without discriminating on the basis of a variable representing membership in a protected class, e.g. ethnicity. In the context of RSs, the work of [20, 22] address a different aspect of fairness (or lack thereof): bias towards popular items. The authors propose a collaborative filtering model which takes into account viewpoints given by users, thereby tackling the tendency for popular items to be recommended more frequently, a problem posed in [29]. A related problem is over-specialization, i.e. the tendency to recommend items similar to those already purchased or liked in the past, which is addressed in [1].

[40] surveys multi-stakeholder RSs, and highlights practical applications. Examples include RSs for sharing economies (e.g. AirBnB, Uber, etc.), online dating [31], and recruiting [38]. [7] discusses fairness in multi-stakeholder RSs, and presents a taxonomy of classes of fairness-aware RSs. The author distinguishes between user fairness, content provider fairness and pairwise fairness, and

Table 1: Consider an arbitrary game, a fixed strategy profile  $\mathbf{X}$  and an arbitrary user  $u_i$ . TOP selects uniformly among the players that satisfy  $u_i$  the most. The Bradley-Terry-Luce mediator [6, 25], or simply BTL, selects player  $j$  w.p. proportional to her satisfaction level over the sum of satisfaction levels. NONE displays no item, and RAND select uniformly among players with a positive satisfaction level. Both TOP and BTL satisfy **F**, but do not satisfy **S**. NONE and RAND satisfy **S**, but do not satisfy **F**. The bottom line refers to the Shapley mediator, SM, which is defined and analyzed in Section 3. In contrast to the other mediators, SM satisfies both **F** and **S**.

MEDIATOR	PROBABILITY COMPUTATION $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$	FAIRNESS ( <b>F</b> )	STABILITY ( <b>S</b> )
TOP	$\frac{\mathbb{1}_{j \in \arg \max_{j'} \sigma_i(X_{j'})}}{ \arg \max_{j'} \sigma_i(X_{j'}) }$	✓	× (THEOREM 1)
BTL	$\frac{\sigma_i(X_j)}{\sum_{j'=1}^N \sigma_i(X_{j'})}$	✓	× (THEOREM 1)
NONE	0	×	✓
RAND	$\frac{\mathbb{1}_{\sigma_i(X_j) > 0}}{\sum_{j'=1}^N \mathbb{1}_{\sigma_i(X_{j'}) > 0}}$	×	✓
SM (SECTION 3)	EQUATION (1)	✓	✓ (THEOREM 2)

reviews applications for these fairness types. A practical problem concerning fairness in multi-stakeholder RSs is discussed in [26]. In their work, an online platform is used by users who play two roles: customers seeking recommendations and content providers aiming for exposure. They report, based on empirical evidence, that collaborative filtering techniques tend to create rich-gets-richer scenarios, and propose a method for re-ranking scores, in order to improve exposure distribution across the content providers.

Note that all the work above considers traditional machine learning tasks that enforce upon the solution some form of fairness, as defined specifically for each task. They suggest additional considerations, but do not consider that the parties (i.e. users, content providers) will change their behavior as a result of the new mechanism, nor examine the game theoretic aspects imposed by the selection of the RS in a formal manner. To the best of our knowledge, our work is the first to suggest a fully grounded approach to content provider fairness in RSs.

Finally, strategic aspects of classical machine learning tasks were also introduced recently [3, 4]. The idea that a recommendation algorithm affects content-provider policy, and as a result must be accompanied by a game-theoretic study is key to recent works in search/information retrieval [2, 33]; so far, however, such work has not dealt with the issue of fairness.

## 2 Problem Formulation

From here on, our ideas will be exemplified in the following motivational example: a mobile application (or simply app) is providing users with valuable content. A set of players (advertisers) publish their items (advertisements) on the app. When a user enters the app, a mediator (RS/advertising engine) decides whether to display an item to that user or not, and which player’s item to display. The reader should notice that while we use that motivation for the purpose of exposition, our formal model and results are applicable to a whole range of RSs with strategic content providers.

Formally, the recommendation game is defined as follows:

- A set of users  $\mathcal{U} = \{u_1, \dots, u_n\}$ , a set of players  $[N] = \{1, \dots, N\}$ , and a mediator  $\mathcal{M}$ .
- The set of items (e.g. possible ad formats/messages to select from) available to player  $j$  is denoted by  $\mathcal{L}_j$ , which we assume to be finite. A *strategy* of player  $j$  is an item from  $\mathcal{L}_j$ .
- Each user  $u_i$  has a satisfaction function  $\sigma_i : \mathcal{L} \rightarrow [0, 1]$ , where  $\mathcal{L} = \bigcup_{j=1}^N \mathcal{L}_j$  is the set of all available items. In general,  $\sigma_i(l)$  measures the *satisfaction level* of  $u_i$  w.r.t.  $l$ .

- When triggered by the app,  $\mathcal{M}$  decides which item to display, if any. Formally, given the strategy profile  $\mathbf{X} = (X_1, \dots, X_N)$  and a user  $u_i$ ,  $\mathcal{M}(\mathbf{X}, u_i)$  is a distribution over  $[N] \cup \{\emptyset\}$ , where  $\emptyset$  symbolizes maintaining the plain content of the app. That is, displaying no item at all. We refer to  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  as the probability that player  $j$ 's item will be displayed to  $u_i$  under the strategy profile  $\mathbf{X}$ .
- Each player gets one monetary unit when her item is displayed to a user; therefore, the expected payoff of player  $j$  under the strategy profile  $\mathbf{X}$  is

$$\pi_j(\mathbf{X}) = \sum_{i=1}^n \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j).$$

- The *social welfare* of the players under the strategy profile  $\mathbf{X}$  is the expected number of displays,

$$V(\mathbf{X}) = \sum_{j=1}^N \pi_j(\mathbf{X}).$$

For ease of notation, we shall sometimes refer to  $\sigma_i(\mathbf{X})$  as the maximum satisfaction level of user  $u_i$  from the items in  $\mathbf{X}$ , i.e.  $\sigma_i(\mathbf{X}) = \max_j \sigma_i(X_j)$ .

We demonstrate our setting with the following example.

**Example 1.** Consider a game with two players and three users. Let  $\mathcal{L}_1 = \{X_1, X_2\}$ ,  $\mathcal{L}_2 = \{X_3\}$  such that the satisfaction levels of the users with respect to the items are

$$\begin{array}{c} \\ \\ \\ \end{array} \begin{array}{ccc} u_1 & u_2 & u_3 \\ \left[ \begin{array}{ccc} 0.1 & 0.9 & 0.2 \\ 0.8 & 0.7 & 0.9 \\ 0.9 & 0.8 & 0.1 \end{array} \right] \end{array}.$$

Consider a mediator displaying each user with the most satisfying item to her taste, denoted by TOP. For example,  $\mathbb{P}(\text{TOP}((X_1, X_3), u_1) = 2) = 1$ , since  $\sigma_1(X_3) = 0.9 > \sigma_1(X_1) = 0.1$ . The profile  $(X_1, X_3)$  will probably be materialized in realistic scenarios, since the payoff of player 1 under the strategy profile  $(X_2, X_3)$  is  $\pi_1(X_2, X_3) = 1$ , while  $\pi_1(X_1, X_3) = 2$ . Notice that from the users' perspective, this profile is not optimal, since  $\sum_{i=1}^3 \sigma_i((X_1, X_3)) = 0.9 + 0.9 + 0.2 = 2$ , while  $\sum_{i=1}^3 \sigma_i((X_2, X_3)) = 2.6$ ; hence, the users suffer from strategic behavior of the players.

After defining general recommendation games, we now present a few properties that one may desire from a mediator. First and foremost, a mediator has to be *fair*. The following is a minimal set of fairness properties.

**Null Player.** If  $\sigma_i(X_j) = 0$ , then it holds that  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = 0$ . Informally, an item will not be displayed to  $u_i$  if it has zero satisfaction level w.r.t. him.

**Symmetry.** If  $u_i$  has the same satisfaction level from two items, they will be displayed with the same probability. Put differently, if  $\sigma_i(X_j) = \sigma_i(X_m)$ , then  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = m)$ .

**User-Independence.** Given the selected items, the display probabilities depends only on the user: if user  $u_{i'}$  is removed from/added to  $\mathcal{U}$ ,  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  will not change, i.e.

$$\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j \mid u_{i'} \in \mathcal{U}).$$

**Leader Monotonicity.**  $\mathcal{M}$  displays the most satisfying items (w.r.t. a specific user) with higher probability than it displays other items. Formally, if  $j \in \arg \max_{j' \in [N]} \sigma_i(X_{j'})$  and  $m \notin \arg \max_{j' \in [N]} \sigma_i(X_{j'})$ , then  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) > \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = m)$ .

For brevity, we denote the above set of fairness properties by **F**. In addition, an essential property in a system with self-motivated participants is that it will be stable. Instability in such systems is a result of a player aiming to improve her payoff given the items selected by others. A minimal requirement in this regard is stability against unilateral deviations as captured by the celebrated pure Nash equilibrium concept, herein denoted PNE.

**Definition 1** (PNE). A strategy profile  $\mathbf{X} = (X_1, \dots, X_N)$  is called a pure Nash equilibrium if for every player  $j \in [N]$  and any strategy  $X'_j \in \mathcal{L}_j$  it holds that  $\pi_j(X_j, \mathbf{X}_{-j}) \geq \pi_j(X'_j, \mathbf{X}_{-j})$ , where  $\mathbf{X}_{-j}$  denotes the vector  $\mathbf{X}$  of all strategies, but with the  $j$ -th component deleted.

We use the notion of PNE to formalize the stability requirement.

**S.** Under any set of players, available items, users and user satisfaction functions, the game induced by  $\mathcal{M}$  possesses a PNE.

The goal of this paper is to devise a computationally tractable mediator that satisfies both **F** and **S**.<sup>1</sup>

## 2.1 Impossibility of Classical Approaches

We highlight a few benchmark mediators in Table 1, including TOP, which was introduced informally in Example 1. Another interesting mediator is BTL, which follows the lines of the Bradley-Terry-Luce model [6, 25]. BTL is addressed here as a representative of a wide family of weight-based mediators: mediators that distribute display probability according to weights, determined by a monotonically increasing function of the satisfaction (e.g., softmax).

Common to TOP, BTL and any other weight-based mediator, is that an item is displayed to a user with probability 1.<sup>2</sup> We model this property as follows.

**Complete.** For any recommendation game and any strategy profile  $\mathbf{X}$ , it holds that  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = 1$ .

Since the goal of an RS is to provide useful content to users, satisfying **Complete** seems justified. Although it seems unreasonable to avoid showing any content to a certain user at a certain time, it turns out that this avoidance is crucial in order to satisfy our requirements.

**Theorem 1.** *No mediator can satisfy **F**, **S** and **Complete**.*

Moreover, Theorem 1 is not sensitive to the sum of the display probabilities,  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$ , being equal to 1. One can show a similar argument for any mediator that displays items with constant probabilities, i.e.  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = c$  for some  $0 \leq c \leq 1$ . Theorem 1 suggests that  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  should bind to the user satisfaction levels. In the next section, we show a novel way of doing so.

## 3 Our Approach: the Shapley Mediator

In order to provide a fair and stable mediator, we resort to cooperative game theory. Informally, a cooperative game consists of two elements: a set of players  $[N]$  and a characteristic function  $v : 2^{[N]} \rightarrow \mathbb{R}$ , where  $v$  determines the value given to every coalition, i.e. every subset of players. The analysis of cooperative games focuses on how the collective payoff of a coalition should be distributed among its members.

One core solution concept in cooperative game theory is the Shapley value [36].

**Definition 2** (Shapley value). Let  $(v, [N])$  be a cooperative game such that  $v(\emptyset) = 0$ . According to the Shapley value, the amount that player  $j$  gets is

$$\frac{1}{N!} \sum_{R \in \Pi([N])} (v(P_j^R \cup \{j\}) - v(P_j^R)),$$

where  $\Pi([N])$  is the set of all permutations of  $[N]$  and  $P_j^R$  is the set of players in  $[N]$  which precede player  $j$  in the permutation  $R$ .

<sup>1</sup>One may require the convergence of any better-response dynamics, thereby allowing the players to learn the environment. In Section 3 we show that our solution satisfies this stronger notion of stability as well.

<sup>2</sup>Perhaps excluding profiles  $\mathbf{X}$  where  $\sigma_i(\mathbf{X}) = 0$ . We allow  $\mathcal{M}$  to behave arbitrarily in this case.

One way to describe the Shapley value, is by imagining the process in which coalitions are formed: when player  $j$  joins coalition  $\mathcal{C}$ , she demands her contribution to the collective payoff of the coalition, namely  $v(\mathcal{C} \cup \{j\}) - v(\mathcal{C})$ . Equation (2) is simply summing over all such possible demands, assuming that all coalitions are equally likely to occur.

For our purposes, we fix a strategy profile  $\mathbf{X}$ , and focus on an arbitrary user  $u_i$ . How should a mediator assign the probabilities of being displayed in a fair fashion? The *induced cooperative game* contains the same set of players. For every  $\mathcal{C} \subseteq [N]$ , let  $\mathbf{X}_{\mathcal{C}}$  denote the strategy profile where all players missing from  $\mathcal{C}$  are removed. We define the characteristic function of the induced cooperative game as

$$v_i(\mathcal{C}; \mathbf{X}) = \sigma_i(\mathbf{X}_{\mathcal{C}}),$$

where  $\sigma_i(\mathbf{X}_{\mathcal{C}})$  is the maximal satisfaction level a user  $u_i$  may obtain from the items chosen by the members of  $\mathcal{C}$ . Indeed, this formulation represents a collaborative behavior of the players, when they aim to maximize the satisfaction of  $u_i$ . Observe that  $v_i(\cdot; \mathbf{X}) : 2^{[N]} \rightarrow \mathbb{R}$  is a valid characteristic function, hence  $(v_i(\cdot; \mathbf{X}), [N])$  is a well defined cooperative game.

Note that the selection of a mediator fully determines the probability of the events  $\mathcal{M}(\mathbf{X}, u_i) = j$ , and vice versa. The mediator that sets the probability of the event  $\mathcal{M}(\mathbf{X}, u_i) = j$  according to the Shapley value of the induced cooperative game  $(v_i(\cdot; \mathbf{X}), [N])$  is hereinafter referred to as *the Shapley mediator*, or SM for abbreviation.

**Proposition 1.** *SM satisfies F.*

We now show that recommendation games with SM possess a PNE. This is done using the notion of potential games [27].

**Definition 3** (Potential games). *A non-cooperative game is called an exact potential game if there exists a function  $\Phi : \prod_j \mathcal{L}_j \rightarrow \mathbb{R}$  such that for any strategy profile  $\mathbf{X} = (X_1, \dots, X_N) \in \prod_j \mathcal{L}_j$ , any player  $j$  and any strategy  $X'_j \in \mathcal{L}_j$ , whenever player  $j$  switches from  $X_j$  to  $X'_j$ , the change in her payoff function equals the change in  $\Phi$ , i.e.*

$$\Phi(X_j, \mathbf{X}_{-j}) - \Phi(X'_j, \mathbf{X}_{-j}) = \pi_j(X_j, \mathbf{X}_{-j}) - \pi_j(X'_j, \mathbf{X}_{-j}).$$

This brings us to the main result of this section.

**Theorem 2.** *Recommendation games with the Shapley mediator are exact potential games.*

Thus, due to [27], any recommendation game with the Shapley mediator possesses at least one PNE, and the set of pure Nash equilibria corresponds to the set of argmax points of the potential function.

**Corollary 1.** *SM satisfies S.*

In fact, Theorem 2 proves a much stronger claim than merely the existence of PNE. A better-response dynamics is a sequential process, where in each iteration an arbitrary player unilaterally deviates to a strategy which increases her payoff.

**Corollary 2.** *In recommendation games with the Shapley mediator, any better-response dynamics converges.*

This convergence guarantee allows the players to learn which items to pick in order to maximize their payoffs. Indeed, as has been observed by work on the topic of online recommendation and advertising systems (e.g. sponsored search [9]), convergence to PNE is essential for system stability, as otherwise inefficient fluctuations may occur.

## 4 Linear Time Implementation

In Section 3 we showed that the Shapley mediator, SM, satisfies **F** and **S**; therefore, it fulfills our requirements stated in Section 2. However, implementation in commercial products would require the mediator to be computationally tractable. The mediator interacts with users, hence a fast response is of great importance. In general, since Equation (2) includes  $2^N$  summands, the computation of the Shapley value in a cooperative game need not be tractable. Indeed, the computation often involves marginal contribution nets [10, 18]. In the following theorem we derive a closed-form formula for calculating the display probabilities under the Shapley mediator, which allows it to compute the display probabilities in linear time.

**Theorem 3.** Let  $\mathbf{X}$  be a strategy profile, and let  $\sigma_i^m(\mathbf{X})$  denote the  $m$ 'th entry in the result of sorting  $(\sigma_i(X_1), \dots, \sigma_i(X_N))$  in ascending order, preserving duplicate elements. The Shapley mediator displays player  $j$ 's item to a user  $u_i$  with probability

$$\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) = \sum_{m=1}^{\rho_i^j(\mathbf{X})} \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N - m + 1}, \quad (1)$$

where  $\sigma_i^0(\mathbf{X}) = 0$ , and  $\rho_i^j(\mathbf{X})$  is an index such that  $\sigma_i(X_j) = \sigma_i^{\rho_i^j(\mathbf{X})}(\mathbf{X})$ .

*Proof.* Denote  $\sigma_i^j = \sigma_i(X_j)$ , and w.l.o.g. let  $\sigma_i^1 \leq \sigma_i^2 \leq \dots \leq \sigma_i^N$ . As defined earlier, the collective payoff of coalition  $\mathcal{C}$  is the maximum satisfaction level of  $u_i$  from an item offered by a player in  $\mathcal{C}$  (the player with the highest index), namely

$$v_i(\mathcal{C}; \mathbf{x}) = \sigma_i(\mathbf{X}_{\mathcal{C}}) = \max_{j \in \mathcal{C}} \sigma_i^j = \sigma_i^{\max \mathcal{C}}.$$

For any permutation  $R$  such that  $\max P_j^R > j$  it follows that  $v_i(P_j^R \cup \{j\}; \mathbf{X}) = v_i(P_j^R; \mathbf{X})$ . Put differently, if in a permutation  $R$  there is a player with index greater than  $j$  that precedes player  $j$ , then player  $j$ 's contribution to the collective payoff of the coalition  $P_j^R \cup \{j\}$  is zero; thus, we ought to look only at permutations where  $\max P_j^R < j$ , and ignore the rest. Denote  $b_j$  as the number of all such permutations. We have

$$b_j = \sum_{m=0}^{j-1} \binom{j-1}{m} m!(N-m-1)!.$$

The latter holds since in every preceding set  $P_j^R$  (prefix of  $R$ ) with  $|P_j^R| = m$ ,  $j$  is located in the  $(m+1)$ -th entry in  $R$ ; thus we have  $\binom{j-1}{m}$  indices to choose from (less than  $j$ ),  $m!$  ways to order them, and another  $(N-m-1)!$  ways to order the suffix (the remaining  $N-m-1$  elements).

For  $r < j$ , we denote by  $a_r$  the number of permutations where  $\max P_j^R = r$ ; hence,

$$a_r = \sum_{m=0}^{r-1} \binom{r-1}{m} (m+1)!(N-m-2)!.$$

Again, we turn to counting arguments: for every prefix  $P_j^R$  of size  $m+1$ , if  $m+1 > r$  then there must be an index greater than  $r$ , therefore  $m \leq r-1$ . Besides  $r$ , there are  $m$  indices in  $P_j^R$ ,  $\binom{r-1}{m}$  ways to choose these indices,  $(m+1)!$  ways to order the prefix, and  $(N-m-2)!$  ways to order the suffix.

**Lemma 1.** It holds that  $b_r = \frac{N!}{N-r+1}$ ,  $a_r = b_{r+1} - b_r$ .

The proof of Lemma 1 is deferred to the appendix. Next, the Shapley value of player  $j$  in the cooperative game  $(v_i(\cdot; \mathbf{X}), [N])$  is

$$\begin{aligned} \phi_j(\sigma_i) &\triangleq \frac{1}{N!} \sum_{R \in \Pi([N])} (v_i(P_j^R \cup \{j\}; \mathbf{X}) - v_i(P_j^R; \mathbf{X})) \\ &= \frac{1}{N!} \sum_{R \in \Pi([N])} (\sigma_i(\mathbf{X}_{P_j^R \cup \{j\}}) - \sigma_i(\mathbf{X}_{P_j^R})). \end{aligned}$$

Since we care only about permutations where the maximum index of a player in the preceding set of player  $j$  is less than  $j$ , we have

$$\phi_j(\sigma_i) = \frac{1}{N!} \sum_{R \in \Pi([N])} \mathbb{1}_{\max P_j^R < j} \left( \sigma_i^j - \sigma_i^{\max P_j^R} \right).$$

Now, using the counting arguments presented above, we derive that

$$\phi_j(\sigma_i) = \frac{1}{N!} \left( b_j \sigma_i^j - \sum_{r=1}^{j-1} a_r \sigma_i^r \right).$$

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**Algorithm 1:** Shapley Mediator

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**Input:** A strategy profile  $\mathbf{X} = (X_1, \dots, X_N)$  and a user  $u_i$

**Output:** An element from  $\{\emptyset, X_1, \dots, X_N\}$

- 1 Pick  $Y$  uniformly at random from  $(0, 1)$
  - 2 **if**  $Y > \max_{j \in [N]} \sigma_i(X_j)$  **then**
  - 3 |   return  $\emptyset$
  - 4 **else**
  - 5 |   Return an element uniformly at random from  $\{X_j \mid j \in [N], \sigma_i(X_j) \geq Y\}$
- 

Applying the concise form of  $a_r$  from Lemma 1 we get

$$\begin{aligned} \phi_j(\sigma_i) &= \frac{1}{N!} \left( \sum_{r=1}^j b_r \sigma_i^r - \sum_{r=1}^{j-1} b_{r+1} \sigma_i^r \right) \\ &= \frac{1}{N!} \left( \sum_{r=1}^j b_r \cdot (\sigma_i^r - \sigma_i^{r-1}) \right) \end{aligned}$$

Ultimately, by using the concise form of  $b$  from Lemma 1 we have

$$\phi_j(\sigma_i) = \sum_{r=1}^j \frac{\sigma_i^r - \sigma_i^{r-1}}{N - r + 1}.$$

□

The Shapley mediator is implemented in Algorithm 1. As an input, it receives a strategy profile and a user, or equivalently user satisfaction levels from that strategy profile. It outputs a player's item with a probability equal to her Shapley value in the cooperative game defined above. Note that the run-time of Algorithm 1 is linear in the number of players, i.e.  $\mathcal{O}(N)$ . A direct result from Theorem 3 and **User-Independence** (see Section 2) is that player payoffs can be calculated efficiently.

**Corollary 3.** *In recommendation games with the Shapley mediator, the payoff of player  $j$  under the strategy profile  $\mathbf{X}$  is given by*

$$\pi_j(\mathbf{X}) = \sum_{i=1}^n \sum_{m=1}^{\rho_i^j(\mathbf{X})} \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N - m + 1}.$$

## 5 Uniqueness of the Shapley Mediator

As analyzed in Subsection 2.1, Theorem 1 suggests that a mediator that sets the probabilities such that  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  is constant cannot satisfy both **F** and **S**. One way of determining  $\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  is defined as follows.

**Efficiency.** The probability of displaying an item to  $u_i$  is the maximal satisfaction level  $u_i$  may obtain from the items chosen in  $\mathbf{X}$ . Formally,

$$\sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = \sigma_i(\mathbf{X}).$$

Efficiency (for brevity, **EF**) binds player payoffs with the maximum satisfaction level of  $u_i$  from the items chosen by the players under  $\mathbf{X}$ . It is well known [15, 36] that the Shapley value is uniquely characterized by properties equivalent to **F** and **EF**, when stated in terms of cooperative games. It is therefore obvious that the Shapley mediator satisfies **EF**;<sup>3</sup> thus, one would expect that the Shapley

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<sup>3</sup>See the proof of Proposition 1 in the appendix. **Leader Monotonicity**, as oppose to the other fairness properties, is not one of Shapley's axioms but rather a byproduct of Shapley's characterization.



mediator will be the only mediator that satisfies **F** and **EF**. This is, however, not the case: consider a mediator that runs TOP w.p.  $\sigma_i(\mathbf{X})$  and NONE otherwise. Clearly, it satisfies **F** and **EF**. In fact, given a mediator  $\mathcal{M}$  satisfying **F** and **Complete**, we can define  $\mathcal{M}'$  such that

$$\mathbb{P}(\mathcal{M}'(\mathbf{X}, u_i) = j) = \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) \cdot \sigma_i(\mathbf{X}), \quad (2)$$

thereby obtaining a mediator satisfying **F** and **EF**.

The question of uniqueness then arises: is **S** derived by satisfying **F** and **EF**? Or even more broadly, are there mediators that satisfy **F**, **S** and **EF** besides the Shapley mediator? Had the answer been yes, this recipe for generating new mediators would have allowed us to seek potentially better mediators, e.g. one satisfying **F**, **S** and **EF** while maximizing user utility. However, as we show next, the Shapley mediator is unique in satisfying **F**, **S** and **EF**.

**Theorem 4.** *The only mediator satisfying **F**, **S** and **EF** is the Shapley mediator.*

*Proof sketch.* Consider an arbitrary mediator  $\mathcal{M}$ , which satisfies **F**, **S** and **EF**. Due to **User-Independence**,  $\mathcal{M}$  sets the display probabilities for a user  $u_i$  according to  $\sigma = (\sigma_i(X_1), \dots, \sigma_i(X_N))$ , which we assume w.l.o.g. to be monotonically non-decreasing; hence, the display probabilities  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j)$  are fully determined by  $\sigma$ , and the problem reduces to showing that  $\mathcal{M}$  must set the display probabilities as SM for every  $\sigma$ . The heart of the theorem relies on showing that unless player  $j$ 's item is displayed with probability  $\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j)$ , she has a beneficial deviation. This implies that  $\mathcal{M}$  is essentially SM.  $\square$

## 6 Player Payoffs and User Utility

In this section we examine the implications of strategic behavior of the players on their payoffs and user utility. Comprehensive treatment of the integration of multiple stakeholders into recommendation calculations was discussed only recently [8], and appears to be challenging. As our work is concerned with strategic content providers, it is natural to consider the Price of Anarchy [24, 35], a common inefficiency measure in non-cooperative games.

### 6.1 Players Payoffs

The Price of Anarchy, herein denoted  $PoA$ , measures the inefficiency in terms of social welfare, as a result of selfish behavior of the players. Specifically, it is the ratio between an optimal dictatorial scenario and the social welfare of the worst PNE. Formally, if  $E_{\mathcal{M}} \subseteq \prod_j \mathcal{L}_j$  is the set of PNE profiles induced by a mediator  $\mathcal{M}$ , then

$$PoA_{\mathcal{M}} = \frac{\max_{\mathbf{X} \in \prod_j \mathcal{L}_j} V(\mathbf{X})}{\min_{\mathbf{X} \in E_{\mathcal{M}}} V(\mathbf{X})} \geq 1.$$

We use the subscript  $\mathcal{M}$  to stress that the  $PoA_{\mathcal{M}}$  depends on the mediator, through the definition of social welfare function  $V$  and player payoffs. Notice that the  $PoA$  of a mediator that does not satisfy **S** can be unbounded, as a PNE may not exist. Quantifying the  $PoA$  can be technically challenging, thus we restrict our analysis to  $PoA_{\text{SM}}$ , the  $PoA$  of the Shapley mediator.

**Theorem 5.**  $PoA_{\text{SM}} \leq \frac{2N-1}{N}$ , and this bound is tight.

Hence, under the Shapley mediator the social welfare of the players can decrease by at most a factor of 2, when compared to an optimal solution.

### 6.2 User Utility

We now examine the implications of using the Shapley mediator on the users. For that, we shall assume that the utility of a user from an item is his satisfaction level from that item. Namely, when item  $l$  is displayed to  $u_i$ , his utility is  $\sigma_i(l)$ . As a result, the expected utility of the users under the strategy profile  $\mathbf{X}$  and a mediator  $\mathcal{M}$  is defined by

$$U_{\mathcal{M}}(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^N \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) \sigma_i(X_j) + \sum_{i=1}^n \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = \emptyset) \sigma_i(\emptyset).$$

Note that the first term results from the displayed items, and the second term from the plain content of the app (displaying no item at all).

To quantify the inefficiency of user utility due to selfish behavior of the players under  $\mathcal{M}$ , we define the *User Price of Anarchy* as

$$UPoA_{\mathcal{M}} = \frac{\max_{\mathcal{M}', \mathbf{X} \in \prod_{j=1}^N \mathcal{L}_j} U_{\mathcal{M}'}(\mathbf{X})}{\min_{\mathbf{X} \in E_{\mathcal{M}}} U_{\mathcal{M}}(\mathbf{X})} \geq 1.$$

The *UPoA* serves as our benchmark for inefficiency of user utility. The nominator is the best possible case: the user utility under any mediator  $\mathcal{M}'$  and any strategy profile  $\mathbf{X}$ . The denominator is the worst user utility under  $\mathcal{M}$ , where  $E_{\mathcal{M}}$  is again the set of PNE profiles induced by  $\mathcal{M}$ . Note that the nominator is independent of  $\mathcal{M}$ .

We first treat users as having zero satisfaction when only the plain content is displayed, i.e.  $\sigma_i(\emptyset) = 0$ , and consider the complementary case afterwards. The following is a negative result for the Shapley mediator.

**Proposition 2.** *The User PoA of the Shapley mediator,  $UPoA_{SM}$ , is unbounded.*

*Proof.* Consider a game with one user and one player with one strategy  $X_1$ , such that  $\sigma_1(X_1) = \epsilon$  for some  $\epsilon > 0$ . It holds that  $U_{SM}(X_1) = \epsilon^2$ , while  $U_{TOP}(X_1) = \epsilon$ ; therefore,  $UPoA_{SM} \geq \frac{1}{\epsilon}$ , which can be arbitrarily large.  $\square$

Proposition 2 questions the applicability of the Shapley mediator. An unavoidable consequence of its use is a potentially destructive effect on user utility. While content-provider fairness is essential, users are the driving force of the RS; therefore, one may advocate for other mediators that perform better with respect to user utility, albeit not necessarily satisfying **S**. Perhaps discarding **S** and adopting a mediator satisfying **Complete** may result in better user utility? Unfortunately, other mediators may lead to a similar decrease in user utility due to strategic behavior of the players, so there appears to be no better solution in this regard.

**Proposition 3.** *The User PoA of TOP,  $UPoA_{TOP}$ , is unbounded.*

*Proof.* Let  $\delta > \epsilon > 0$  be arbitrarily small, and consider the satisfaction matrix

$$\begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cc} u_1 & u_2 \\ \left[ \begin{array}{cc} 1 & 0 \\ \delta & \delta \\ \epsilon & \epsilon \end{array} \right] \end{array}.$$

Let  $\mathcal{L}_1 = \{X_1, X_2\}$  and  $\mathcal{L}_2 = \{X_3\}$ . Under TOP, the only PNE is  $(X_2, X_3)$  with  $U_{TOP}(X_2, X_3) = 2\delta$ , but  $U_{TOP}(X_1, X_3) = 1 + \epsilon$ ; therefore,  $UPoA_{TOP} = \frac{1+\epsilon}{2\delta}$ , which can be arbitrarily large.  $\square$

Using similar arguments, one can show that  $UPoA_{BTL}$  is unbounded as well.

In many situations it is reasonable to assume that when no item is displayed to a user, his utility is 1. Namely,  $\sigma_i(\emptyset) = 1$  for every user  $u_i$ . Indeed, this seems aligned with the ads-in-apps model: the user is interrupted when an advertisement is displayed. We refer to this scenario as the *optimal plain content case*.

From here on, we adopt this perspective for upper-bounding the *UPoA*. Observe that user utility is therefore maximized when no item is displayed whatsoever. Nevertheless, displaying no item will also result in zero payoff for the players. Here too,  $UPoA_{TOP}$  is unbounded, while  $UPoA_{NONE} = 1$ . The following lemma bounds user utility under the Shapley mediator.

**Lemma 2.** *In the optimal plain content case, it holds for any recommendation game and any strategy profile  $\mathbf{X}$  that  $U_{SM}(\mathbf{X}) \geq \frac{n}{4}$ .*

In fact, numerical calculations show that for a practical number of players (up to 50,000) the user utility is greater than  $0.568n$ . See the appendix for further discussion. Finally,

**Corollary 4.** *In the optimal plain content case, it holds that  $UPoA_{SM} \leq 4$ .*

## 7 Discussion

Our results are readily extendable in the following important direction (which is even further elaborated in the appendix). In many online scenarios, content providers typically customize the items they offer to accommodate specific individuals. Indeed, personalization is applied in a variety of fields in order to improve user satisfaction. Specifically, consider the case where each player may promote a set of items, where different items may be targeted towards different users, and the size of this set is determined exogenously (e.g., by her budget). In this case, a player selects a set of items which she then provides to the mediator. Here the Shapley mediator satisfies  $\mathbf{F}$  and  $\mathbf{S}$ , the game induced by the Shapley mediator is still a potential game, and the computation of the Shapley mediator still takes linear time.

Theorem 4 shows that the only mediator satisfying  $\mathbf{F}$ ,  $\mathbf{S}$  and  $\mathbf{EF}$  is the Shapley mediator. One future research objective is to determine whether the Shapley mediator is fully characterized by  $\mathbf{F}$ ,  $\mathbf{S}$  and if these two properties be satisfied by another mediator. Obviously, probabilistic selection between the Shapley mediator and one of  $\{\text{NONE}, \text{RAND}\}$  (see Table 1) would work, but are those the only options? Theorem 4 disqualifies mediators satisfying  $\mathbf{EF}$ . In addition, Theorem 1 implies that **Complete** cannot be satisfied simultaneously with  $\mathbf{F}$ ,  $\mathbf{S}$ .

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## A Omitted proofs

### A.1 Proof of Theorem 1

*Proof.* Consider the following satisfaction matrix:

$$\begin{array}{c} u_1 \quad u_2 \quad u_3 \\ a \begin{bmatrix} 0 & y & x \\ x & 0 & y \\ y & x & 0 \end{bmatrix}, \\ b \\ c \end{array}$$

where  $(x, y) \in (0, 1]^2$ . Let  $\mathcal{L}_1 = \mathcal{L}_2 = \{a, b, c\}$  (i.e. symmetric two-player game). This recommendation game induces a normal form game. Since  $\mathcal{M}$  satisfies **User-Independence**, it sets the display probability as a function of the satisfaction levels only (i.e. regardless of the identity of the user); therefore, to facilitate writing the proof, we denote

$$\mathcal{M}(\sigma_i(X_1), \sigma_i(X_2))_1 \triangleq \mathbb{P}(\mathcal{M}((X_1, X_2), u_i) = 1).$$

Namely,  $\mathcal{M}(\sigma_i(X_1), \sigma_i(X_2))_1$  is the probability that player 1's item will be displayed to  $u_i$  under the strategy profile  $(X_1, X_2)$ . Since  $\mathcal{M}$  satisfies **Complete** and **Symmetry**,

$$\mathcal{M}(\sigma_i(X_1), \sigma_i(X_2))_1 = 1 - \mathcal{M}(\sigma_i(X_1), \sigma_i(X_2))_2 = \mathcal{M}(\sigma_i(X_2), \sigma_i(X_1))_2.$$

Since  $\mathcal{M}$  satisfies both **Complete** and **Null Player**, one has to define what happens if the satisfaction vector is the zero vector (see Footnote 2). Denote  $\mathcal{M}(0, 0)_1 = \alpha$ , and due to **Symmetry**  $\mathcal{M}(0, 0)_2 = \alpha$  also (notice that  $\alpha \leq 0.5$ ). The following matrix describes the payoff of player 1 under any possible strategy profile:

$$\begin{array}{c} a \quad b \quad c \\ a \begin{bmatrix} \alpha + \mathcal{M}(x, x)_1 + \mathcal{M}(y, y) & \mathcal{M}(0, x)_1 + \mathcal{M}(y, 0)_1 + \mathcal{M}(x, y)_1 & \mathcal{M}(0, y)_1 + \mathcal{M}(y, x)_1 + \mathcal{M}(x, 0)_1 \\ \mathcal{M}(x, 0)_1 + \mathcal{M}(0, y)_1 + \mathcal{M}(y, x)_1 & \alpha + \mathcal{M}(x, x)_1 + \mathcal{M}(y, y) & \mathcal{M}(x, y)_1 + \mathcal{M}(0, x)_1 + \mathcal{M}(y, 0)_1 \\ \mathcal{M}(y, 0)_1 + \mathcal{M}(x, y)_1 + \mathcal{M}(0, x)_1 & \mathcal{M}(y, x)_1 + \mathcal{M}(x, 0)_1 + \mathcal{M}(0, y)_1 & \alpha + \mathcal{M}(x, x)_1 + \mathcal{M}(y, y) \end{bmatrix}, \\ b \\ c \end{array}$$

Denote  $\beta = \mathcal{M}(0, x)_1 + \mathcal{M}(y, 0)_1 + \mathcal{M}(x, y)_1$ . Due to **Symmetry**, the game is described by the following bi-matrix:

$$\begin{array}{c} a \quad b \quad c \\ a \begin{bmatrix} 1 + \alpha, 1 + \alpha & \beta, 3 - \beta & 3 - \beta, \beta \\ 3 - \beta, \beta & 1 + \alpha, 1 + \alpha & \beta, 3 - \beta \\ \beta, 3 - \beta & 3 - \beta, \beta & 1 + \alpha, 1 + \alpha \end{bmatrix}. \\ b \\ c \end{array}$$

Clearly, since  $\alpha \leq 0.5$ , this game possesses a PNE only if  $\beta = 1.5$ . Otherwise, under any strategy profile there exists a player with a beneficial deviation; hence,

$$\beta = \mathcal{M}(0, x)_1 + \mathcal{M}(y, 0)_1 + \mathcal{M}(x, y)_1 = 1.5.$$

Due to **Null Player**, we have  $\mathcal{M}(0, x)_1 = 0$ ,  $\mathcal{M}(y, 0)_1 = 1$ , therefore  $\mathcal{M}(x, y)_1 = 0.5$  for every  $(x, y) \in (0, 1]^2$ , and **Leader Monotonicity** does not hold.  $\square$

### A.2 Proof of Proposition 1

*Proof.* Denote the Shapley value of player  $j$  in a cooperative game  $(v, [N])$  by

$$\phi_j(v) = \frac{1}{N!} \sum_{R \in \Pi([N])} (v(P_j^R \cup \{j\}) - v(P_j^R)). \quad (3)$$

It is well known (see, e.g., [15, 36]) that the Shapley value satisfies the following properties:

- C1. **Null Player:** If  $v(\mathcal{C} \cup \{j\}) = v(\mathcal{C})$  for every coalition  $\mathcal{C} \subseteq [N]$ , then  $\phi_j(v) = 0$ .
- C2. **Symmetry:** If  $v(\mathcal{C} \cup \{j\}) = v(\mathcal{C} \cup \{m\})$  for every coalition  $\mathcal{C} \subseteq [N] \setminus \{j, m\}$ , then  $\phi_j(v) = \phi_m(v)$ .

C3. Linearity: If  $(v, [N])$  and  $(w, [N])$  are two cooperative games and  $\lambda$  is a real number, it holds that

$$\phi_j(v + \lambda w) = \phi_j(v) + \lambda \phi_j(w).$$

C4. Efficiency:  $\sum_{j \in [N]} \phi_j(v) = v([N])$ .

Note that properties C1-C4 are akin to **F** and **EF** defined in Sections 2 and 5 respectively. However, C1-C4 are properties of the Shapley value in *cooperative game*  $(v, [N])$ , while **F**, **EF** refer to recommendation games; hence, we ought to show that the Shapley mediator, defined as the Shapley value in the cooperative game  $v_i(\cdot; \mathbf{X})$ , satisfies **F** and **EF**. Denote by  $\phi_j(v_i(\cdot; \mathbf{X}))$  the Shapley value of player  $j$  in  $(v_i(\cdot; \mathbf{X}), [N])$ , where  $v_i(\mathcal{C}; \mathbf{X}) = \sigma_i(\mathbf{X}_{\mathcal{C}})$ . Recall that by definition of the Shapley mediator,

$$\phi_j(v_i(\cdot; \mathbf{X})) = \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j). \quad (4)$$

**Null Player.** If  $\sigma_i(X_j) = 0$ ,

$$v_i(\mathcal{C} \cup \{j\}; \mathbf{X}) = \max_{j' \in \mathcal{C} \cup \{j\}} \sigma_i(X_{j'}) = \max_{j' \in \mathcal{C}} \sigma_i(X_{j'}) = v_i(\mathcal{C}; \mathbf{X});$$

therefore,

$$\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) \stackrel{\text{Eq(4)}}{=} \phi_j(v_i(\cdot; \mathbf{X})) \stackrel{\text{C1}}{=} 0.$$

**Symmetry.** If  $\sigma_i(X_j) = \sigma_i(X_m)$ , for every  $\mathcal{C} \subseteq [N] \setminus \{j, m\}$  it holds that

$$\begin{aligned} v_i(\mathcal{C} \cup \{j\}; \mathbf{X}) &= \max_{j' \in \mathcal{C} \cup \{j\}} \sigma_i(X_{j'}) = \begin{cases} \sigma_i(X_j) & \sigma_i(X_j) \geq \max_{j' \in \mathcal{C}} \sigma_i(X_{j'}) \\ \max_{j' \in \mathcal{C}} \sigma_i(X_{j'}) & \text{Otherwise} \end{cases} \\ &= \begin{cases} \sigma_i(X_m) & \sigma_i(X_m) \geq \max_{j' \in \mathcal{C}} \sigma_i(X_{j'}) \\ \max_{j' \in \mathcal{C}} \sigma_i(X_{j'}) & \text{Otherwise} \end{cases} \\ &= \max_{j' \in \mathcal{C} \cup \{m\}} \sigma_i(X_{j'}) = v_i(\mathcal{C} \cup \{m\}; \mathbf{X}). \end{aligned} \quad (5)$$

Therefore,

$$\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) = \phi_j(v_i(\cdot; \mathbf{X})) \stackrel{\text{C2}}{=} \phi_m(v_i(\cdot; \mathbf{X})) = \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = m).$$

**User-Independence.** Notice that  $\phi_j(v_i(\cdot; \mathbf{X}))$  is solely determined by  $(\sigma_i(X_1), \dots, \sigma_i(X_N))$ ; therefore

$$\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = \phi_j(v_i(\cdot; \mathbf{X})) = \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j \mid u_i \in \mathcal{U}).$$

**Leader Monotonicity.** Let  $j \in \arg \max_{j' \in [N]} \sigma_i(X_{j'})$  and  $m \notin \arg \max_{j' \in [N]} \sigma_i(X_{j'})$ , and hence  $\sigma_i(X_j) > \sigma_i(X_m)$ . Given a permutation  $R$  over the elements of  $[N]$ , define  $R(j \leftrightarrow m)$  to be the same permutation vector where  $j$  and  $m$  are swapped. Notice that if  $j$  precedes  $m$  in  $R$ , then  $v_i(P_j^R) = v_i(P_m^{R(j \leftrightarrow m)})$ , thus

$$v_i(P_j^R \cup \{j\}; \mathbf{X}) - v_i(P_j^R; \mathbf{X}) \geq v_i(P_m^{R(j \leftrightarrow m)} \cup \{m\}; \mathbf{X}) - v_i(P_m^{R(j \leftrightarrow m)}; \mathbf{X}). \quad (6)$$

Alternatively, if  $m$  precedes  $j$  in  $R$  (and therefore  $j$  precedes  $m$  in  $R(j \leftrightarrow m)$ ) we have

$$v_i(P_m^{R(j \leftrightarrow m)} \cup \{m\}; \mathbf{X}) = v_i(P_m^{R(j \leftrightarrow m)}; \mathbf{X}),$$

$$v_i(P_j^R \cup \{j\}; \mathbf{X}) - v_i(P_j^R; \mathbf{X}) \geq v_i(P_m^{R(j \leftrightarrow m)} \cup \{m\}; \mathbf{X}) - v_i(P_m^{R(j \leftrightarrow m)}; \mathbf{X}) = 0. \quad (7)$$

In addition, for  $R^*$  in which  $j$  appears in the first entry, it holds that

$$\sigma_i(X_j) = v_i(P_j^{R^*} \cup \{j\}; \mathbf{X}) - v_i(P_j^{R^*}; \mathbf{X}) > v_i(P_m^{R^*(j \leftrightarrow m)} \cup \{m\}; \mathbf{X}) - v_i(P_m^{R^*(j \leftrightarrow m)}; \mathbf{X}) = \sigma_i(X_m). \quad (8)$$

Finally, due to Equations (6),(7) and (8), by summing over all permutations and dividing by  $\frac{1}{N!}$  we get

$$\begin{aligned}
\phi_j(v_i(\cdot; \mathbf{X})) &\stackrel{\text{Def.}}{=} \frac{1}{N!} \sum_{R \in \Pi([N])} (v_i(P_j^R \cup \{j\}; \mathbf{X}) - v_i(P_j^R; \mathbf{X})) \\
&> \frac{1}{N!} \sum_{R(j \leftrightarrow m) \in \Pi([N])} (v_i(P_j^{R(j \leftrightarrow m)} \cup \{j\}; \mathbf{X}) - v_i(P_j^{R(j \leftrightarrow m)}; \mathbf{X})) \\
&= \frac{1}{N!} \sum_{R \in \Pi([N])} (v_i(P_m^R \cup \{m\}; \mathbf{X}) - v_i(P_m^R; \mathbf{X})) \\
&= \phi_m(v_i(\cdot; \mathbf{X})), \tag{9}
\end{aligned}$$

hence  $\mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) > \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = m)$ .

**EF.** We have

$$\sum_{j \in [N]} \mathbb{P}(\mathcal{M}(\mathbf{X}, u_i) = j) = \sum_{j \in [N]} \phi_j(v_i(\cdot; \mathbf{X})) \stackrel{\text{C4}}{=} v_i([N]; \mathbf{X}) \stackrel{\text{Def.}}{=} \sigma_i(\mathbf{X}).$$

This concludes the proof of the proposition.  $\square$

### A.3 Proof of Theorem 2

*Proof.* We prove Theorem 2 by showing that recommendation games with the Shapley mediator (denoted RGSM for brevity) belong to the class of congestion games [34]. Due to [27], this implies that RGSM are potential games. A congestion game is a non-cooperative game, defined by players and resources, where the payoff of each player depends solely on the resources she chooses and on the number of players that chose each of the corresponding resources. Formally, a congestion game is a tuple  $([N], \mathcal{R}, (S_j)_{j \in [N]}, (w_r)_{r \in \mathcal{R}})$  where:

- $[N]$  is the set of players.
- $\mathcal{R}$  is the set of resources.
- $S_j$  denotes the set of possible strategies of player  $j$ , where any  $s_j \in S_j$  is a subset of all resources.
- The number of players who select resource  $r$  under the strategy profile  $\mathbf{s} = (s_1, \dots, s_N)$  is given by  $k_r(\mathbf{s}) = |\{j : r \in s_j\}|$
- $w_r$  is a utility function,  $w_r : \mathbb{N} \rightarrow \mathbb{R}_+$ , such that  $w_r(k)$  is given to any player whose strategy contains resource  $r$ , in case exactly  $k$  players chose  $r$ .
- The payoff of player  $j$  under the strategy profile  $\mathbf{s}$  is given by  $\sum_{r \in s_j} w_r(k_r(\mathbf{s}))$ .

Given an RGSM game instance, we construct a corresponding congestion game, and show that the payoffs of the players under any strategy profile is exactly the same in both games. Importantly, the complexity of the below reduction is irrelevant, and the reduction is presented only to assure the existence of a PNE in our analyzed class of games.

Denote

$$E = \{\sigma_i(l) : i \in [n], l \in \mathcal{L}\} \cup \{0, 1\},$$

and observe that  $B \triangleq |E| - 1 \leq n|\mathcal{L}| - 1 < \infty$ . Let  $\epsilon_0, \epsilon_1, \dots, \epsilon_B$  be the ordered elements of  $E$  in ascending order. Next, we aim to represent a strategy profile as a selection of corresponding resources by the players. Denote  $\mathcal{R} = \{r_m^i : m = 1 \dots B, i = 1 \dots n\}$ , where for each user  $u_i$  resource  $r_m^i$  is associated with the interval  $[\epsilon_{m-1}, \epsilon_m]$ . The strategy of selecting item  $l \in \mathcal{L}$  is modeled as selecting all resources associated with intervals that are subsets of  $[0, \sigma_i(l)]$ , namely

$$\mathcal{A}(l) = \{r_m^i : \sigma_i(l) \geq \epsilon_m, m \in [B], i \in [n]\};$$



thus, there is an induced one-to-one function from the set of items to the power set of resources,  $\mathcal{A} : \mathcal{L} \rightarrow 2^{\mathcal{R}}$ . Mapping between items and resources, we define the set of possible strategies of player  $j$ :

$$S_j = \{\mathcal{A}(l) : l \in \mathcal{L}_j\}.$$

The load of each resource (the number of players that select this resource) under the strategy profile  $\mathbf{X} = (X_1, \dots, X_N)$  is:

$$k_m^i(\mathcal{A}(\mathbf{X})) = |\{j : r_m^i \in \mathcal{A}(X_j)\}|,$$

where  $\mathcal{A}(\mathbf{X}) = (\mathcal{A}(X_1), \dots, \mathcal{A}(X_N))$ . The utility of selecting resource  $r_m^i$  depends only on the number of players who select it, and is defined as follows:

$$w_m^i(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dots & \dots \\ \frac{\epsilon_m - \epsilon_{m-1}}{k} & \text{if } x = k \\ \dots & \dots \\ \frac{\epsilon_m - \epsilon_{m-1}}{N} & \text{if } x = N \end{cases}.$$

Thus the congestion game  $([N], \mathcal{R}, (S_j)_{j \in [N]}, (w_r)_{r \in \mathcal{R}})$  is properly defined. The remaining ingredient of the reduction is:

**Lemma 3.** *The sum of utilities of each player  $j$  in the congestion game under the strategy profile  $\mathcal{A}(\mathbf{X})$  is exactly her payoff in the RGSM under the strategy profile  $\mathbf{X}$ :*

$$\pi_j(\mathbf{X}) = \sum_{\substack{i, m: \\ r_m^i \in \mathcal{A}(X_j)}} w_m^i(k_m^i(\mathcal{A}(\mathbf{X}))).$$

*Proof.* Fix a user  $u_i$ . Recall that  $\sigma_i^m(\mathbf{X})$  is the  $m$ 'th satisfaction level  $u_i$  obtains from the items in  $\mathbf{X}$  (in ascending order,  $\sigma_i^0(\mathbf{X}) = 0$ ), and  $\rho_i^j(\mathbf{X})$  is the number of items offered to  $u_i$  with a satisfaction level less or equal to  $\sigma_i^j(X_j)$ .

Under the strategy profile  $\mathbf{X}$ , define  $\epsilon_{\alpha_1}, \dots, \epsilon_{\alpha_N}$  to be the elements in  $E$  such that  $\epsilon_{\alpha_m} = \sigma_i^m(\mathbf{X})$ . Observe that for any  $m \in [N]$  the number of players that selected resources associated with intervals contained in  $[\epsilon_{\alpha_{m-1}}, \epsilon_{\alpha_m}]$  is  $N - m + 1$ . In addition, let  $\alpha(m)$  be the index such that  $\epsilon_{\alpha(m)} = \sigma_i^m(\mathbf{X})$ .

The strategy  $X_j$  of player  $j$  is mapped to the set of resources  $\mathcal{A}(X_j)$ , therefore

$$\begin{aligned} \sum_{m=1}^B w_m^i(k_m^i(\mathcal{A}(\mathbf{X}))) \mathbb{1}_{r_m^i \in \mathcal{A}(X_j)} &= \sum_{m=1}^{\alpha(j)} w_m^i(k_m^i(\mathcal{A}(\mathbf{X}))) \\ &= \sum_{m=1}^{\alpha(j)} \frac{\epsilon_m - \epsilon_{m-1}}{k_m^i(\mathcal{A}(\mathbf{X}))} = \sum_{m=1}^{\rho_i^j(\mathbf{X})} \frac{\epsilon_{\alpha_m} - \epsilon_{\alpha_{m-1}}}{N - m + 1} \\ &= \sum_{m=1}^{\rho_i^j(\mathbf{X})} \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N - m + 1} = \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j). \end{aligned}$$

By summing over all users we get

$$\begin{aligned} \sum_{i=1}^n \sum_{m=1}^B w_m^i(k_m^i(\mathcal{A}(\mathbf{X}))) \mathbb{1}_{r_m^i \in \mathcal{A}(X_j)} \\ = \sum_{i=1}^n \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) = \pi_j(\mathbf{X}). \end{aligned}$$

This concludes the proof of Lemma 3.  $\square$

Ultimately, since RGSM belong to the class of congestion games, they possess a potential function, and every better response dynamic converges [27]. This Concludes the proof of Theorem 2.  $\square$

In fact, using standard arguments, we can show that

$$\Phi(\mathbf{X}) = \sum_{r_m^i \in \mathcal{R}} \sum_{j=1}^{k_m^i(\mathcal{A}(\mathbf{X}))} w_m^i(j)$$

is the potential function of RGSM.

#### A.4 Proof of Lemma 1

*Proof.* First we show that  $b_r = \frac{N!}{N-r+1}$ :

$$\begin{aligned} b_j &= \sum_{m=0}^{j-1} \binom{j-1}{m} m!(N-m-1)! \\ &= \sum_{m=0}^{j-1} \left[ \frac{(N-j)!}{(N-j)!} \frac{(j-1)!}{m!(j-m-1)!} m!(N-m-1)! \right] \\ &= (N-j)!(j-1)! \sum_{m=0}^{j-1} \left[ \frac{(N-m-1)!}{(N-j)!(j-m-1)!} \right] \\ &= (N-j)!(j-1)! \sum_{m=0}^{j-1} \binom{N-m-1}{N-j} \\ &\stackrel{m'=N-m-1}{=} (N-j)!(j-1)! \sum_{m'=N-1}^{N-j} \binom{m'}{N-j} \\ &\stackrel{r=N-j}{=} (N-j)!(j-1)! \sum_{m'=r}^{N-1} \binom{m'}{r}. \end{aligned} \tag{10}$$

By Chu Shih-Chieh's Identity (see e.g. [12]) it holds that

$$\sum_{i=r}^k \binom{i}{r} = \binom{k+1}{r+1}. \tag{11}$$

Applying Equation (11) to Equation (10) with  $r = N-j, k = N-1$  we get:

$$\begin{aligned} b_j &= (N-j)!(j-1)! \binom{N}{N-j+1} \\ &= (N-j)!(j-1)! \frac{N!}{(N-j+1)!(j-1)!} \\ &= \frac{N!}{N-j+1}. \end{aligned}$$

In addition,  $a_r = b_{r+1} - b_r$  since:

$$\begin{aligned} b_{r+1} - b_r &= \sum_{m=0}^r \binom{r}{m} m!(N-m-1)! \\ &\quad - \sum_{m=0}^{r-1} \binom{r-1}{m} m!(N-m-1)! \\ &= \sum_{m=0}^r m!(N-m-1)! \left[ \binom{r}{m} - \binom{r-1}{m} \right] \\ &= \sum_{m=1}^r m!(N-m-1)! \left[ \binom{r}{m} - \binom{r-1}{m} \right] \end{aligned}$$

Using Pascal's rule

$$\binom{r}{m} = \binom{r-1}{m-1} + \binom{r-1}{m},$$

we get

$$\begin{aligned} b_{r+1} - b_r &= \sum_{m=1}^r m!(N-m-1)! \binom{r-1}{m-1} \\ &= \sum_{m=0}^r (m+1)!(N-m-2)! \binom{r-1}{m} = a_r. \end{aligned}$$

□

### A.5 Proof of Theorem 4

The proof of Theorem 4 relies on several supporting lemmas.

Due to **User-Independence**,  $\mathcal{M}$  sets the display probabilities according to the satisfaction vector only; thus, it is enough to show that  $\mathcal{M}$  satisfying **F, S** must distribute the display probabilities exactly as the Shapley mediator does for any satisfaction vector  $\sigma$ . Since we do not have a specific user in mind, we denote by  $\mathcal{M}(\sigma)_i$  the probability that  $\mathcal{M}$  will display an arbitrary user the item of player  $i$  under the satisfaction vector  $\sigma$ . Namely, if the strategy profile  $\mathbf{X}$  induces a satisfaction vector  $\sigma$  for user  $u_j$  we denote  $\mathcal{M}(\sigma)_i = \mathbb{P}(\mathcal{M}(\mathbf{X}, u_j) = i)$ . Let  $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^N)$ , and w.l.o.g. let  $\sigma^1 \leq \sigma^2 \leq \dots \leq \sigma^N$ . Let  $\sigma^{-i}$  denote the vector  $\sigma$  with the  $i$ -th component deleted.

**Observation 1.** *If  $\sigma$  contains one non-zero entry  $i$ , then  $\mathcal{M}(\sigma)_i = \text{SM}(\sigma)_i$  for every player  $i$ .*

This observation follows immediately from **Null Player** and **EF**. Next, we show that in a two player game, each player gets her Shapley value.

**Lemma 4.** *Let  $N = 2$ . For every  $(\sigma^1, \sigma^2) \in \mathbb{R}^2$  and every  $i \in \{1, 2\}$ , it holds that  $\mathcal{M}(\sigma^1, \sigma^2)_i = \text{SM}(\sigma^1, \sigma^2)_i$ .*

*Proof.* Assume by contradiction that  $\mathcal{M}(\sigma^1, \sigma^2) \neq \text{SM}(\sigma^1, \sigma^2)$ . First, we analyze the case  $\mathcal{M}(\sigma^1, \sigma^2) = (\frac{\sigma^1}{2} + \epsilon, \sigma^2 - \frac{\sigma^1}{2} - \epsilon)$  for some  $\epsilon > 0$ . Consider the following satisfaction matrix:

$$\begin{array}{c} \begin{array}{ccc} & u_1 & u_2 & u_3 \\ x_1 & \left[ \begin{array}{ccc} \sigma^1 & 0 & 0 \\ 0 & 0 & \sigma^1 + \frac{\epsilon}{2} \\ \sigma^2 & 0 & 0 \\ 0 & \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2} & 0 \end{array} \right] \end{array} \end{array},$$

with  $\mathcal{L}_1 = \{x_1, y_1\}$  and  $\mathcal{L}_2 = \{x_2, y_2\}$ . This satisfaction matrix induces the following  $2 \times 2$  normal-form game

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{c} x_2 \\ y_2 \end{array} \\ \begin{array}{c} x_1 \\ y_1 \end{array} & \left[ \begin{array}{cc} \frac{\sigma^1}{2} + \epsilon, \sigma^2 - \frac{\sigma^1}{2} - \epsilon & \sigma^1, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2} \\ \sigma^1 + \frac{\epsilon}{2}, \sigma^2 & \sigma^1 + \frac{\epsilon}{2}, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2} \end{array} \right] \end{array} \end{array},$$

since

$$\begin{aligned} (\pi(x_1, x_2)_1, \pi(x_1, x_2)_2) &= (\mathcal{M}(\sigma^1, \sigma^2)_1, \mathcal{M}(\sigma^1, \sigma^2)_2) = \left( \frac{\sigma^1}{2} + \epsilon, \sigma^2 - \frac{\sigma^1}{2} - \epsilon \right) \\ (\pi(x_1, y_2)_1, \pi(x_1, y_2)_2) &= \left( \mathcal{M}(\sigma^1, 0)_1, \mathcal{M}\left(0, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2}\right)_2 \right) = \left( \sigma^1, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2} \right) \\ (\pi(y_1, y_2)_1, \pi(y_1, y_2)_2) &= \left( \mathcal{M}\left(\sigma^1 + \frac{\epsilon}{2}, 0\right)_1, \mathcal{M}\left(0, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2}\right)_2 \right) = \left( \sigma^1 + \frac{\epsilon}{2}, \sigma^2 - \frac{\sigma^1}{2} - \frac{\epsilon}{2} \right) \\ (\pi(y_1, x_2)_1, \pi(y_1, x_2)_2) &= \left( \mathcal{M}\left(\sigma^1 + \frac{\epsilon}{2}, 0\right)_1, \mathcal{M}(0, \sigma^2)_2 \right) = \left( \sigma^1 + \frac{\epsilon}{2}, \sigma^2 \right). \end{aligned}$$

This  $2 \times 2$  normal-form game contains a cycle of beneficial deviations, which implies the non-existence of PNE, hence we obtain a contradiction. On the other hand, let  $\mathcal{M}(\sigma^1, \sigma^2) = (\frac{\sigma^1}{2} - \epsilon, \sigma^2 - \frac{\sigma^1}{2} + \epsilon)$  for some  $\epsilon > 0$ . Consider the following satisfaction matrix:

$$\begin{array}{c} x_1 \\ y_1 \\ x_2 \\ y_2 \end{array} \begin{array}{ccc} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \\ \left[ \begin{array}{ccc} \sigma^1 & 0 & 0 \\ 0 & \sigma^1 - \epsilon & 0 \\ \sigma^2 & \sigma^1 - \epsilon & 0 \\ 0 & 0 & \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4} \end{array} \right] \end{array}$$

where again  $\mathcal{L}_1 = \{x_1, y_1\}$  and  $\mathcal{L}_2 = \{x_2, y_2\}$ . This satisfaction matrix induces the following  $2 \times 2$  normal-form game:

$$\begin{array}{c} x_1 \\ y_1 \end{array} \left[ \begin{array}{cc} \begin{array}{c} x_2 \\ y_2 \end{array} \\ \left( \begin{array}{cc} \frac{\sigma^1}{2} - \epsilon, \sigma^2 + \frac{\sigma^1}{2} & \sigma^1, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4} \\ \frac{\sigma^1}{2} - \frac{\epsilon}{2}, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{2} & \sigma^1 - \epsilon, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4} \end{array} \right) \end{array} \right],$$

since

$$\begin{aligned} (\pi(x_1, x_2)_1, \pi(x_1, x_2)_2) &= (\mathcal{M}(\sigma^1, \sigma^2)_1, \mathcal{M}(\sigma^1, \sigma^2)_2 + \mathcal{M}(\sigma^1 - \epsilon, 0)_2) = \left( \frac{\sigma^1}{2} - \epsilon, \sigma^2 - \frac{\sigma^1}{2} + \epsilon + \sigma^1 - \epsilon \right) \\ &= \left( \frac{\sigma^1}{2} - \epsilon, \sigma^2 + \frac{\sigma^1}{2} \right) \\ (\pi(y_1, x_2)_1, \pi(y_1, x_2)_2) &= (\mathcal{M}(\sigma^1 - \epsilon, \sigma^1 - \epsilon)_1, \mathcal{M}(0, \sigma^2)_2 + \mathcal{M}(\sigma^1 - \epsilon, \sigma^1 - \epsilon)_2) \\ &= \left( \frac{\sigma^1}{2} - \frac{\epsilon}{2}, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{2} \right) \\ (\pi(y_1, y_2)_1, \pi(y_1, y_2)_2) &= \left( \mathcal{M}(\sigma^1 - \epsilon, 0)_1, \mathcal{M}\left(0, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4}\right)_2 \right) \\ &= \left( \sigma^1 - \epsilon, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4} \right) \\ (\pi(x_1, y_2)_1, \pi(x_1, y_2)_2) &= \left( \mathcal{M}(\sigma^1, 0)_1, \mathcal{M}\left(0, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4}\right)_2 \right) \\ &= \left( \sigma^1, \sigma^2 + \frac{\sigma^1}{2} - \frac{\epsilon}{4} \right). \end{aligned}$$

Again we obtained a contradiction to satisfying **S**. Overall,  $\mathcal{M}$  must produce the same distribution as SM for  $N = 2$ .  $\square$

Since any mediator behaves like Shapley mediator when  $N = 2$ , due to **Null Player** it also holds for  $N \geq 2$  for satisfaction vector  $\sigma$  with only two non-zero entries.

**Lemma 5.** Let  $0 \leq \sigma^{1'} \leq \sigma^1 \leq \sigma^2 \leq \dots \leq \sigma^N$ . It holds that  $\mathcal{M}(\sigma^{-1}, \sigma^{1'})_1 + \frac{\sigma^1 - \sigma^{1'}}{N} = \mathcal{M}(\sigma)_1$ .

*Proof.* The assertion holds for  $\sigma^{1'} = \sigma^1$ . Otherwise, we prove the assertion by induction, where Lemma 4 serves as the base case. Assume the claim holds for  $N - 1$ , and does not hold for  $N$ .

*Case 1:*  $\mathcal{M}(\sigma)_1 - \mathcal{M}(\sigma^{-1}, \sigma^{1'})_1 - \frac{\sigma^1 - \sigma^{1'}}{N} > \epsilon > 0$ . Due to **EF**,

$$\begin{aligned} \sum_{i=1}^N (\mathcal{M}(\sigma^{-1}, \sigma^{1'})_i - \mathcal{M}(\sigma)_i) &= 0 \\ \Rightarrow \sum_{i=2}^N (\mathcal{M}(\sigma^{-1}, \sigma^{1'})_i - \mathcal{M}(\sigma)_i) &= \mathcal{M}(\sigma)_1 - \mathcal{M}(\sigma^{-1}, \sigma^{1'})_1 > \frac{\sigma^1 - \sigma^{1'}}{N}. \\ \Rightarrow \max_i (\mathcal{M}(\sigma^{-1}, \sigma^{1'})_i - \mathcal{M}(\sigma)_i) &> \frac{\sigma^1 - \sigma^{1'}}{N(N-1)}. \end{aligned} \tag{12}$$

Denote by  $i$  a player such that

$$\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \mathcal{M}(\boldsymbol{\sigma})_i > \frac{\sigma^1 - \sigma^{1'}}{N(N-1)}. \quad (13)$$

Consider the following satisfaction matrix

$$\begin{array}{c} x_1 \\ y_1 \\ x_i \\ y_i \\ x_j \end{array} \left[ \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \sigma^{1'} & \sigma^1 - \sigma^{1'} & \epsilon & 0 \\ \sigma^1 & 0 & 0 & 0 \\ \sigma^i & \sigma^1 - \sigma^{1'} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} - \epsilon \\ \sigma^j & \sigma^1 - \sigma^{1'} & 0 & 0 \end{array} \right],$$

where  $\mathcal{L}_1 = \{x_1, y_1\}$ ,  $\mathcal{L}_i = \{x_i, y_i\}$  and  $\mathcal{L}_j = \{x_j\}$  for every player  $j$  such that  $j \notin \{1, i\}$ . Note that all players but 1 and  $i$  are non-strategic, or alternatively every strategy they select have the same satisfaction levels w.r.t. users  $\{u_1, u_2, u_3, u_4\}$ . We have the following cycle:

- $(\pi(x_1, x_i)_1, \pi(x_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_1 + \frac{\sigma^1 - \sigma^{1'}}{N} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} \right)$ .
- $(\pi(y_1, x_i)_1, \pi(y_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma})_1, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} \right)$ . Due to Equation (13), we have  $\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} - \mathcal{M}(\boldsymbol{\sigma})_i - \frac{\sigma^1 - \sigma^{1'}}{N-1} > \frac{\sigma^1 - \sigma^{1'}}{N} - \frac{\sigma^1 - \sigma^{1'}}{N-1} + \frac{\sigma^1 - \sigma^{1'}}{N(N-1)} = 0$ .
- $(\pi(y_1, y_i)_1, \pi(y_1, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_1, \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} - \epsilon \right)$ . According to the inductive step,  $\mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_1 = \mathcal{M}(\boldsymbol{\sigma}^{-\{1, i\}}, \sigma^{1'}, 0)_1 + \frac{\sigma^1 - \sigma^{1'}}{N-1}$ .
- $(\pi(x_1, y_i)_1, \pi(x_1, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-\{1, i\}}, \sigma^{1'}, 0)_1 + \frac{\sigma^1 - \sigma^{1'}}{N-1} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} - \epsilon \right)$ .
- $(\pi(x_1, x_i)_1, \pi(x_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_1 + \frac{\sigma^1 - \sigma^{1'}}{N} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} \right)$

The reader can verify that each step the deviating player (e.g. player 1 from the first bullet to the second, and player  $i$  from the second to the third) has indeed a beneficial deviation; hence we have a cycle, and a PNE does not exist, which is a contradiction to satisfying **S**.

Case 2:  $\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_1 + \frac{\sigma^1 - \sigma^{1'}}{N} - \mathcal{M}(\boldsymbol{\sigma})_1 > \epsilon > 0$ . Similarly to the previous case,

$$\begin{aligned} \sum_{i=1}^N (\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= 0 \\ \sum_{i=2}^N (\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= \mathcal{M}(\boldsymbol{\sigma})_1 - \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_1 < \frac{\sigma^1 - \sigma^{1'}}{N} \\ \Rightarrow \min_i (\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &< \frac{\sigma^1 - \sigma^{1'}}{N(N-1)} \end{aligned} \quad (14)$$

Denote by  $i$  a player such that

$$\mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \mathcal{M}(\boldsymbol{\sigma})_i < \frac{\sigma^1 - \sigma^{1'}}{N(N-1)} \Leftrightarrow \mathcal{M}(\boldsymbol{\sigma})_i - \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i > -\frac{\sigma^1 - \sigma^{1'}}{N(N-1)}. \quad (15)$$

Consider the following game:

$$\begin{array}{c} x_1 \\ y_1 \\ x_i \\ y_i \\ x_j \end{array} \left[ \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \sigma^1 & 0 & \epsilon & 0 \\ \sigma^{1'} & \sigma^1 - \sigma^{1'} & 0 & 0 \\ \sigma^i & \sigma^1 - \sigma^{1'} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} - \epsilon \\ \sigma^j & \sigma^1 - \sigma^{1'} & 0 & 0 \end{array} \right],$$

where  $\mathcal{L}_1 = \{x_1, y_1\}$ ,  $\mathcal{L}_i = \{x_i, y_i\}$  and  $\mathcal{L}_j = \{x_j\}$  for every player  $j$  such that  $j \notin \{1, i\}$ . Here again all players but 1 and  $i$  are non-strategic, or alternatively every strategy they select have the same satisfaction levels w.r.t. users  $\{u_1, u_2, u_3, u_4\}$ . We have the following cycle:

- $(\pi(x_1, x_i)_1, \pi(x_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma})_1 + \epsilon, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} \right)$
- $(\pi(y_1, x_i)_1, \pi(y_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_1 + \frac{\sigma^1 - \sigma^{1'}}{N}, \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i + \frac{\sigma^1 - \sigma^{1'}}{N} \right)$ . Due to Equation (15)

$$\mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} - \mathcal{M}(\boldsymbol{\sigma}^{-1}, \sigma^{1'})_i - \frac{\sigma^1 - \sigma^{1'}}{N} > 0.$$

- $(\pi(y_1, y_i)_1, \pi(y_1, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-\{1, i\}}, \sigma^{1'}, 0)_1 + \frac{\sigma^1 - \sigma^{1'}}{N-1}, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} - \epsilon \right)$ . According to the inductive step,

$$\mathcal{M}(\boldsymbol{\sigma}^{-\{1, i\}}, \sigma^{1'}, 0)_1 + \frac{\sigma^{1'} - \sigma^1}{N-1} = \mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_1.$$

- $(\pi(x_1, y_i)_1, \pi(x_1, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_1 + \epsilon, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} - \epsilon \right)$
- $(\pi(x_1, x_i)_1, \pi(x_1, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma})_1 + \epsilon, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^1 - \sigma^{1'}}{N-1} \right)$ .

Hence we have a cycle, which is a contradiction to satisfying **S**. This concludes the proof of this lemma.  $\square$

**Corollary 5.** Given  $\boldsymbol{\sigma}$ , by setting  $\sigma^{1'} = 0$  and relying on **Null Player**, we get  $\mathcal{M}(\boldsymbol{\sigma})_1 = \text{SM}(\boldsymbol{\sigma})_1 = \frac{\sigma^1}{N}$ .

**Lemma 6.** Let  $0 \leq \sigma^{1'} \leq \sigma^1 \leq \dots \leq \sigma^{k'} \leq \sigma^k \leq \dots \leq \sigma^{N'} \leq \sigma^N \leq 1$ . If  $\mathcal{M}$  satisfies **F**, **S** and **EF**, it holds for every player index  $k$  that

$$\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N - k + 1} = \mathcal{M}(\boldsymbol{\sigma})_k.$$

*Proof.* We prove the claim by induction over  $k$  and  $N$ , player index and number of players respectively.

Base cases:

1. The assertion holds for  $k \in \{1, 2\}$  and  $N = 2$  due to Lemma 4.
2. The assertion holds for  $k = 1$  and  $N \geq 1$  Due to Corollary 5.

Assume the assertion holds for  $(j, N)$  and  $(j, N - 1)$ , for every  $j < k$ . Next, we show it holds for  $(k, N)$ , for  $k < N$ , while dealing with  $k = N$  afterwards.

*Case 1:* Assume by contradiction that  $\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N - k + 1} < \mathcal{M}(\boldsymbol{\sigma})_k$ . **EF** suggests the sum of probabilities remains  $\sigma^N$ , and due to the inductive step we have  $\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_j = \mathcal{M}(\boldsymbol{\sigma})_j$  for  $j < k$ ; hence,

$$\begin{aligned} \sum_{i=1}^N (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= 0 \\ \sum_{i=k+1}^N (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= \mathcal{M}(\boldsymbol{\sigma})_k - \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k > \frac{\sigma^k - \sigma^{k'}}{N - k + 1}. \\ \Rightarrow \max_i (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &> \frac{\sigma^k - \sigma^{k'}}{(N - k)(N - k + 1)}. \end{aligned} \quad (16)$$

Denote by  $i$  a player such that

$$\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i > \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)}, \quad (17)$$

and let  $\epsilon = \min\{\epsilon_k, \epsilon_i\}$  for  $\epsilon_k, \epsilon_i$  that satisfy

$$\mathcal{M}(\boldsymbol{\sigma})_k - \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k - \frac{\sigma^k - \sigma^{k'}}{N-k+1} > \epsilon_k > 0, \quad \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i - \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)} > \epsilon_i > 0.$$

Consider the following game

$$\begin{array}{c} x_k \\ y_k \\ x_i \\ y_i \\ x_r \\ x_m \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \left[ \begin{array}{cccc} \sigma^{k'} & \sigma^k - \sigma^{k'} & \epsilon & 0 \\ \sigma^k & 0 & 0 & 0 \\ \sigma^i & \sigma^k - \sigma^{k'} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N-k+1} - \epsilon \\ \sigma^r & 0 & 0 & 0 \\ \sigma^m & \sigma^k - \sigma^{k'} & 0 & 0 \end{array} \right],$$

where  $\mathcal{L}_k = \{x_k, y_k\}$ ,  $\mathcal{L}_i = \{x_i, y_i\}$ ,  $\mathcal{L}_r = \{x_r\}$  for  $1 \leq r < k$  and  $\mathcal{L}_m = \{x_m\}$  for  $k < m \leq N$ . Note that all players but  $k$  and  $i$  are non-strategic, or alternatively every strategy they select have the same satisfaction levels w.r.t. users  $\{u_1, u_2, u_3, u_4\}$ . We have the following cycle:

- $(\pi(x_k, x_i)_k, \pi(x_k, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N-k+1} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N-k+1} \right)$ .
- $(\pi(y_k, x_i)_k, \pi(y_k, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma})_k, \mathcal{M}(\boldsymbol{\sigma})_i + \frac{\sigma^k - \sigma^{k'}}{N-k} \right)$ . Due to Equation (17), we have  $\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N-k+1} - \mathcal{M}(\boldsymbol{\sigma})_i - \frac{\sigma^k - \sigma^{k'}}{N-k} > \frac{\sigma^k - \sigma^{k'}}{N-k+1} - \frac{\sigma^k - \sigma^{k'}}{N-k} + \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)} = 0$ .
- $(\pi(y_k, y_i)_k, \pi(y_k, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_k, \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N-k+1} - \epsilon \right)$ . According to the inductive step,  $\mathcal{M}(\boldsymbol{\sigma}^{-i}, 0)_k = \mathcal{M}(\boldsymbol{\sigma}^{-\{k,i\}}, \sigma^{k'}, 0)_k + \frac{\sigma^k - \sigma^{k'}}{N-k}$ .
- $(\pi(x_k, y_i)_k, \pi(x_k, y_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-\{k,i\}}, \sigma^{k'}, 0)_k + \frac{\sigma^k - \sigma^{k'}}{N-k} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N} - \epsilon \right)$ .
- $(\pi(x_k, x_i)_k, \pi(x_k, x_i)_i) = \left( \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N-k+1} + \epsilon, \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N-k+1} \right)$ .

Hence we have a cycle, which is a contradiction to satisfying **S**.

*Case 2:*  $\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N-k+1} > \mathcal{M}(\boldsymbol{\sigma})_k$ . Due to **EF**,

$$\begin{aligned} \sum_{i=1}^N (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= 0 \\ \sum_{i=k+1}^N (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &= \mathcal{M}(\boldsymbol{\sigma})_k - \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_k < \frac{\sigma^k - \sigma^{k'}}{N-k+1}. \\ \Rightarrow \min_i (\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i) &< \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)} \end{aligned} \quad (18)$$

Denote by  $i$  a player such that

$$\mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i - \mathcal{M}(\boldsymbol{\sigma})_i < \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)} \Leftrightarrow \mathcal{M}(\boldsymbol{\sigma})_i - \mathcal{M}(\boldsymbol{\sigma}^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{(N-k)(N-k+1)} > 0, \quad (19)$$

and let  $\epsilon = \min\{\epsilon_k, \epsilon_i\}$  for  $\epsilon_k, \epsilon_i$  that satisfy

$$\mathcal{M}(\sigma^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N - k + 1} - \mathcal{M}(\sigma)_k > \epsilon_k > 0, \quad \mathcal{M}(\sigma)_i - \mathcal{M}(\sigma^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{(N - k)(N - k + 1)} > \epsilon_i > 0.$$

Consider the following game:

$$\begin{array}{c} x_k \\ y_k \\ x_i \\ y_i \\ x_r \\ x_m \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \left[ \begin{array}{cccc} \sigma^k & 0 & \epsilon & 0 \\ \sigma^{k'} & \sigma^k - \sigma^{k'} & 0 & 0 \\ \sigma^i & \sigma^k - \sigma^{k'} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} - \epsilon \\ \sigma^r & 0 & 0 & 0 \\ \sigma^m & \sigma^k - \sigma^{k'} & 0 & 0 \end{array} \right],$$

where again  $\mathcal{L}_k = \{x_k, y_k\}$ ,  $\mathcal{L}_i = \{x_i, y_i\}$ ,  $\mathcal{L}_r = \{x_r\}$  for  $1 \leq r < k$  and  $\mathcal{L}_m = \{x_m\}$  for  $k < m \leq N$ . We have the following cycle:

- $(\pi(x_k, x_i)_k, \pi(x_k, x_i)_i) = \left( \mathcal{M}(\sigma)_k + \epsilon, \mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} \right)$ .
- $(\pi(y_k, x_i)_k, \pi(y_k, x_i)_i) = \left( \mathcal{M}(\sigma^{-k}, \sigma^{k'})_k + \frac{\sigma^k - \sigma^{k'}}{N - k + 1}, \mathcal{M}(\sigma^{-k}, \sigma^{k'})_i + \frac{\sigma^k - \sigma^{k'}}{N - k + 1} \right)$ . Due to Equation (19)

$$\mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} - \mathcal{M}(\sigma^{-k}, \sigma^{k'})_i - \frac{\sigma^k - \sigma^{k'}}{N - k + 1} > 0.$$

- $(\pi(y_k, y_i)_k, \pi(y_k, y_i)_i) = \left( \mathcal{M}(\sigma^{-\{k, i\}}, \sigma^{k'}, 0)_k + \frac{\sigma^k - \sigma^{k'}}{N - k}, \mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} - \epsilon \right)$ . According to the inductive step,

$$\mathcal{M}(\sigma^{-\{k, i\}}, \sigma^{k'}, 0)_k + \frac{\sigma^{k'} - \sigma^k}{N - k} = \mathcal{M}(\sigma^{-i}, 0)_k.$$

- $(\pi(x_k, y_i)_k, \pi(x_k, y_i)_i) = \left( \mathcal{M}(\sigma^{-i}, 0)_k + \epsilon, \mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} - \epsilon \right)$
- $(\pi(x_k, x_i)_k, \pi(x_k, x_i)_i) = \left( \mathcal{M}(\sigma)_k + \epsilon, \mathcal{M}(\sigma)_i + \frac{\sigma^k - \sigma^{k'}}{N - k} \right)$

Hence we obtained a cycle, which is a contradiction to satisfying **S**. The only missing ingredient is the case of  $k = N$ . Due to Lemma 5 it holds that  $\mathcal{M}(\sigma)_1 = \text{SM}(\sigma)_1$ , thus due to **Symmetry** we have

$$\mathcal{M}(\sigma)_2 = \mathcal{M}(\sigma^{-2}, \sigma^1)_2 + \frac{\sigma^2 - \sigma^1}{N - 1} = \text{SM}(\sigma^{-2}, \sigma^1)_2 + \frac{\sigma^2 - \sigma^1}{N - 1} = \text{SM}(\sigma)_2.$$

Using this technique we obtain  $\mathcal{M}(\sigma)_k = \text{SM}(\sigma)_k$  for every player  $k < N$ . Finally, due to **EF**,

$$\mathcal{M}(\sigma)_N = \sigma^N - \sum_{k=1}^{N-1} \mathcal{M}(\sigma)_k = \sigma^N - \sum_{k=1}^{N-1} \text{SM}(\sigma)_k \Rightarrow \mathcal{M}(\sigma)_N = \text{SM}(\sigma)_N. \quad (20)$$

Since the Shapley mediator satisfies the assertion, we have

$$\mathcal{M}(\sigma)_N = \mathcal{M}(\sigma^{-N}, \sigma^{N'})_N + \frac{\sigma^N - \sigma^{N'}}{1}$$

as well. This concludes the proof of the lemma.  $\square$

*Proof of Theorem 4.* The proof of Theorem 4 Follows directly from Lemma 6.  $\square$



## A.6 Proof of Theorem 5

*Proof.* Recall that under the Shapley mediator it holds that

$$V(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^N \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) = \sum_{i=1}^n \sigma_i(\mathbf{X}).$$

The following analysis holds for every  $\sigma_i$  as defined in the model. For ease of notation, we handle the functions  $\sigma_i$  and  $V$  as set functions. Namely, for  $L \subseteq \mathcal{L}$  let

$$\sigma_i(L) = \max_{l \in L} \sigma_i(l),$$

and

$$V(L) = \sum_{i=1}^n \sigma_i(L) = \sum_{i=1}^n \max_{l \in L} \sigma_i(l).$$

In addition, recall that a strategy profile  $\mathbf{X}$  is often referred to as the set of items selected by the players. Next, we lower-bound the payoff of a player as a function of the social welfare.

**Lemma 7.** *For every strategy profile  $\mathbf{X}$  it holds that*

$$\pi_j(\mathbf{X}) \geq \frac{V(X_j)}{N} + \frac{N-1}{N} (V(\mathbf{X}) - V(\mathbf{X}_{-j})).$$

*Proof.* For a user  $u_i$ , exactly one of the following holds:

- If  $\sigma_i(\mathbf{X}_{-j}) \geq \sigma_i(X_j)$ , then  $\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) \geq \frac{1}{N} \sigma_i(X_j)$  holds, with equality when player  $j$  offers the least satisfying item to  $u_i$ .
- If  $\sigma_i(\mathbf{X}_{-j}) < \sigma_i(X_j)$ , player  $j$  gets the full difference between the terms:  $\sigma_i(X_j) - \sigma_i(\mathbf{X}_{-j})$ , as well as at least  $\frac{1}{N}$  times  $\sigma_i(\mathbf{X}_{-j})$ .

These cases are disjoint, thus:

$$\begin{aligned} \pi_j(\mathbf{X}) &= \sum_{i=1}^n \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) \\ &= \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ \geq \sigma_i(X_j)}} \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) + \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ < \sigma_i(X_j)}} \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) \\ &\geq \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ \geq \sigma_i(X_j)}} \frac{\sigma_i(X_j)}{N} + \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ < \sigma_i(X_j)}} \left( \sigma_i(X_j) - \sigma_i(\mathbf{X}_{-j}) + \frac{\sigma_i(\mathbf{X}_{-j})}{N} \right) \\ &= \frac{1}{N} \sum_{i=1}^n \sigma_i(X_j) + \frac{N-1}{N} \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ < \sigma_i(X_j)}} (\sigma_i(X_j) - \sigma_i(\mathbf{X}_{-j})). \end{aligned}$$

Notice that if  $\sigma_i(\mathbf{X}_{-j}) < \sigma_i(X_j)$  it holds that  $\sigma_i(\mathbf{X}) - \sigma_i(\mathbf{X}_{-j}) = \sigma_i(X_j) - \sigma_i(\mathbf{X}_{-j})$ , and if  $\sigma_i(\mathbf{X}_{-j}) \geq \sigma_i(X_j)$  then  $\sigma_i(\mathbf{X}) - \sigma_i(\mathbf{X}_{-j}) = 0$ . As a result,

$$\begin{aligned} &= \frac{1}{N} \sum_{i=1}^n \sigma_i(X_j) + \frac{N-1}{N} \sum_{\substack{i: \sigma_i(\mathbf{X}_{-j}) \\ < \sigma_i(X_j)}} \sigma_i(\mathbf{X}) - \sigma_i(\mathbf{X}_{-j}) \\ &\geq \frac{1}{N} \sum_{i=1}^n \sigma_i(X_j) + \frac{N-1}{N} \sum_{i=1}^n \sigma_i(\mathbf{X}) - \sigma_i(\mathbf{X}_{-j}) \\ &= \frac{V(X_j)}{N} + \frac{N-1}{N} (V(\mathbf{X}) - V(\mathbf{X}_{-j})) \end{aligned}$$

which concludes the proof of this lemma.  $\square$

One more necessary definition is the following:

**Definition 4** (Submodular function). *We say that  $f : 2^\Omega \rightarrow \mathbb{R}$  is submodular if for any  $X, Y \in \Omega$  such that  $X \subseteq Y$  and every  $x \in \Omega \setminus Y$  it holds that -*

$$f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$$

Note that by definition of  $\sigma_i$  it is a monotonically increasing set function. In addition,

**Lemma 8.**  $\sigma_i$  is submodular.

*Proof.* For arbitrary  $i, j, \mathbf{X}$  and  $l \in \mathcal{L}$ , we need to show that

$$\sigma_i(\mathbf{X}_{-j} \cup \{l\}) - \sigma_i(\mathbf{X}_{-j}) \geq \sigma_i(\mathbf{X} \cup \{l\}) - \sigma_i(\mathbf{X}) \quad (21)$$

If  $\sigma_i(\{l\}) \leq \sigma_i(\mathbf{X})$ , then by monotonicity the right hand side of Equation (21) equals zero while the left hand side is non-negative. Alternatively, if  $\sigma_i(\{l\}) > \sigma_i(\mathbf{X})$  then  $\sigma_i(\mathbf{X} \cup \{l\}) = \sigma_i(\mathbf{X}_{-j} \cup \{l\})$ . Moreover,  $\sigma_i(\mathbf{X}) \geq \sigma_i(\mathbf{X}_{-j})$ , hence

$$\sigma_i(\mathbf{X}_{-j} \cup \{l\}) - \sigma_i(\mathbf{X}_{-j}) = \sigma_i(\mathbf{X} \cup \{l\}) - \sigma_i(\mathbf{X}_{-j}) \geq \sigma_i(\mathbf{X} \cup \{l\}) - \sigma_i(\mathbf{X})$$

□

By summing Equation (21) over all users, we get:

**Corollary 6.** *The social welfare function  $V$  is submodular.*

We are now ready to prove the theorem: denote the optimal solution as  $\mathbf{X}^*$  and an arbitrary PNE profile  $\mathbf{X}$ . Since  $\mathbf{X}$  is in equilibrium it follows that  $\pi_j(\mathbf{X}) \geq \pi_j(X_j^*, \mathbf{X}_{-j})$ ; therefore,

$$V(\mathbf{X}) = \sum_{j=1}^N \pi_j(\mathbf{X}) \geq \sum_{j=1}^N \pi_j(X_j^*, \mathbf{X}_{-j}). \quad (22)$$

Due to Lemma 7 we have:

$$\pi_j(X_j^*, \mathbf{X}_{-j}) \geq \frac{V(X_j^*)}{N} + \frac{N-1}{N} (V(X_j^*, \mathbf{X}_{-j}) - V(\mathbf{X}_{-j})). \quad (23)$$

Summing Equation (23) over all players:

$$\sum_{j=1}^N \pi_j(X_j^*, \mathbf{X}_{-j}) \geq \sum_{j=1}^N \frac{V(X_j^*)}{N} + \frac{N-1}{N} \sum_{j=1}^N (V(X_j^*, \mathbf{X}_{-j}) - V(\mathbf{X}_{-j})). \quad (24)$$

Observe that due to submodularity, for every  $j$  it holds that

$$V(X_j^*, \mathbf{X}_{-j}) - V(\mathbf{X}_{-j}) \geq V(X_1^*, \dots, X_{j-1}^*, X_j^*, \mathbf{X}) - V(X_1^*, \dots, X_{j-1}^*, \mathbf{X}).$$

Thus:

$$\begin{aligned} \sum_{j=1}^N (V(X_j^*, \mathbf{X}_{-j}) - V(\mathbf{X}_{-j})) &\geq \sum_{j=1}^N (V(X_1^*, \dots, X_{j-1}^*, X_j^*, \mathbf{X}) - V(X_1^*, \dots, X_{j-1}^*, \mathbf{X})) \\ &= V(\mathbf{X}^*, \mathbf{X}) - V(\mathbf{X}) \\ &\geq V(\mathbf{X}^*) - V(\mathbf{X}). \end{aligned} \quad (25)$$

By substituting Equation (25) into Equation (24) we get:

$$\begin{aligned} V(\mathbf{X}) &\geq \sum_{j=1}^N \frac{V(X_j^*)}{N} + \frac{N-1}{N} (V(\mathbf{X}^*) - V(\mathbf{X})) \\ &\geq \frac{V(\mathbf{X}^*)}{N} + \frac{N-1}{N} (V(\mathbf{X}^*) - V(\mathbf{X})) \\ &= V(\mathbf{X}^*) - \frac{N-1}{N} V(\mathbf{X}) \end{aligned} \quad (26)$$

Finally, by Equations (22) and (26) we have  $V(\mathbf{X}) \geq V(\mathbf{X}^*) - \frac{N-1}{N}V(\mathbf{X})$ , and

$$PoA \triangleq \frac{V(\mathbf{X}^*)}{V(\mathbf{X})} \leq \frac{2N-1}{N}.$$

After upper-bounding the  $PoA$ , our objective is to show a game instance which achieves this bound. Consider a symmetric  $N$ -player game with  $N$  users, and items  $\mathcal{L} = \mathcal{L}_j = \{l_1, \dots, l_N, l^*\}$ . In addition:

$$\forall i \in \{1, \dots, N\}: \quad \sigma_i(x) = \begin{cases} 1 & x = l_i \\ a & x = l^* \\ 0 & \text{otherwise} \end{cases}$$

The optimal social welfare is obtained when each player selects a unique item (e.g. player  $j$  selects  $l_j$ ). In that case, the Shapley mediator will display an item to every user with probability 1, hence the social welfare is  $N$ . Observe that the strategy profile  $\mathbf{X} = (l^*, l^*, \dots, l^*)$  is in equilibrium: consider the payoff of player  $j$  under  $\mathbf{X}$ , and a possible unilateral deviation to  $l_i$ :

$$\pi_j(\mathbf{X}) = \sum_{i=1}^N \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) = N \frac{a}{N} = a, \quad \pi_j(l_i, \mathbf{X}_{-j}) = \frac{a}{N} + (1-a).$$

For  $a = \frac{N}{2N-1}$  we get  $\pi_j(\mathbf{X}) = \pi_j(l_i, \mathbf{X}_{-j})$ , therefore  $\mathbf{X}$  is an equilibrium profile. Overall,

$$PoA = \frac{N}{aN} = \frac{1}{a} = \frac{2N-1}{N}.$$

This concludes the proof of Theorem 5. □

## A.7 Proof of Lemma 2

*Proof.* Fix an arbitrary user  $u_i$  and a strategy profile  $\mathbf{X}$ , and denote

$$j^* \in \min \left\{ j : \sigma_i^j(\mathbf{X}) > \frac{\sigma_i^N(\mathbf{X})}{2} \right\}.$$

Observe that

$$\begin{aligned} & \sum_{j=1}^N (\sigma_i(X_j) \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j)) + \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = \emptyset) \\ &= \sum_{j=1}^N \sigma_i^j(\mathbf{X}) \left( \sum_{m=1}^j \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N-m+1} \right) + 1 - \sigma_i^N(\mathbf{X}) \\ &\geq \sum_{j=j^*}^N \sigma_i^j(\mathbf{X}) \left( \sum_{m=1}^j \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N-m+1} \right) + 1 - \sigma_i^N(\mathbf{X}) \\ &\geq \sum_{j=j^*}^N \sigma_i^{j^*}(\mathbf{X}) \left( \sum_{m=1}^j \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N-m+1} \right) + 1 - \sigma_i^N(\mathbf{X}) \\ &\geq \frac{\sigma_i^N(\mathbf{X})}{2} \sum_{j=j^*}^N \sum_{m=1}^j \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N-m+1} + 1 - \sigma_i^N(\mathbf{X}) \\ &\geq \frac{\sigma_i^N(\mathbf{X})}{2} \left( \sigma_i^N(\mathbf{X}) - \sigma_i^{j^*-1}(\mathbf{X}) \right) + 1 - \sigma_i^N(\mathbf{X}) \\ &\geq \frac{\sigma_i^N(\mathbf{X})}{2} \left( \sigma_i^N(\mathbf{X}) - \frac{\sigma_i^N(\mathbf{X})}{2} \right) + 1 - \sigma_i^N(\mathbf{X}) \\ &= \frac{\sigma_i^N(\mathbf{X})}{2} \frac{\sigma_i^N(\mathbf{X})}{2} + 1 - \sigma_i^N(\mathbf{X}) \end{aligned} \tag{27}$$

The minimum value obtained by the function  $y = \frac{x^2}{4} - x + 1$  in the segment  $x \in [0, 1]$  is  $\frac{1}{4}$ . Finally, Equation (27) holds for every user, hence

$$U_{\text{SM}}(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^N (\sigma_i(X_j) \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j)) + \sum_{i=1}^n \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = \emptyset) \geq \frac{n}{4}.$$

□

## B On User Utility

As mentioned in Section 6, numerical calculations show that the lower bound of  $\frac{n}{4}$  on user utility is far from tight. We present here the methods employed to obtain the tighter (numerical) bound.

Recall that under the Shapley mediator:

$$\begin{aligned} U_{\text{SM}}(\mathbf{X}) &= \sum_{i=1}^n \sum_{j=1}^N (\mathbb{P}(\text{SM}(\mathbf{X}, u_i) = j) \sigma_i(X_j)) + \sum_{i=1}^n \mathbb{P}(\text{SM}(\mathbf{X}, u_i) = \emptyset) \\ &= \sum_{i=1}^n \sum_{j=1}^N \sigma_i^j(\mathbf{X}) \left( \sum_{m=1}^j \frac{\sigma_i^m(\mathbf{X}) - \sigma_i^{m-1}(\mathbf{X})}{N - m + 1} \right) + \sum_{i=1}^n (1 - \sigma_i^N(\mathbf{X})). \end{aligned}$$

Due to linearity, user utility cannot be less than  $n$  times the minimum utility one user can get; hence, we focus on a game with one user. In addition, despite  $U_{\text{SM}}$  being a function of the strategy profile, it is more convenient to present it as a function of the satisfaction levels  $\boldsymbol{\sigma} = (\sigma^1, \dots, \sigma^N)$  without the need to state the strategy profile; this can be done due to **User-Independence**. We have:

$$U_{\text{SM}}(\boldsymbol{\sigma}) = \sum_{j=1}^N \sigma^j \sum_{m=1}^j \frac{\sigma^m - \sigma^{m-1}}{N - j + 1} + (1 - \sigma^N). \quad (28)$$

Note that

$$\begin{aligned} \sum_{m=1}^j \frac{\sigma^m - \sigma^{m-1}}{N - m + 1} &= \sum_{m=1}^j \frac{\sigma^m}{N - m + 1} - \sum_{m=1}^j \frac{\sigma^{m-1}}{N - m + 1} \\ &= \sum_{m=1}^j \frac{\sigma^m}{N - m + 1} - \sum_{m=1}^{j-1} \frac{\sigma^m}{N - m} \\ &= \frac{\sigma^j}{N - j + 1} - \sum_{m=1}^{j-1} \frac{\sigma^m}{(N - m)(N - m + 1)}. \end{aligned}$$

Hence  $U_{\text{SM}}(\boldsymbol{\sigma})$  can be presented as

$$\begin{aligned} U_{\text{SM}}(\boldsymbol{\sigma}) &= \sum_{j=1}^N \sigma^j \left( \frac{\sigma^j}{N - j + 1} - \sum_{m=1}^{j-1} \frac{\sigma^m}{(N - m)(N - m + 1)} \right) + (1 - \sigma^N) \\ \Rightarrow U_{\text{SM}}(\boldsymbol{\sigma}) &= \sum_{j=1}^N \frac{\sigma^j \sigma^j}{N - j + 1} - \sum_{j=1}^N \sigma^j \sum_{m=1}^{j-1} \frac{\sigma^m}{(N - m)(N - m + 1)} + (1 - \sigma^N); \end{aligned}$$

therefore,

$$\frac{dU_{\text{SM}}}{d\sigma^j}(\boldsymbol{\sigma}) = \begin{cases} \frac{2\sigma^j}{N-j+1} - \sum_{m=1}^{j-1} \frac{\sigma^m}{(N-m)(N-m+1)} - \sum_{m=j+1}^N \frac{\sigma^m}{(N-j)(N-j+1)} & j < N \\ 2\sigma^N - \sum_{m=1}^{N-1} \frac{\sigma^m}{(N-m)(N-m+1)} - 1 & j = N \end{cases}.$$

Thus, by taking the partial derivatives to zero, one can find the argmin point (more precisely, the argmin vector) of  $U_{\text{SM}}$ . We solved this system of linear equations numerically for various value of  $N$ . The results are presented in Figure 1. Note that  $U_{\text{SM}}(\boldsymbol{\sigma}) > 0.568$  for up to 50,000 players. Going back to games with arbitrary number of users, we have  $U_{\text{SM}}(\mathbf{X}) > 0.568n$  for every strategy profile  $\mathbf{X}$ .

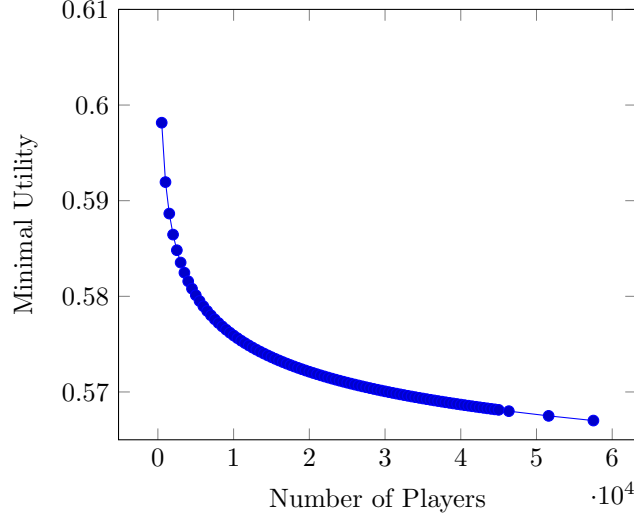


Figure 1: Numerical calculation of the minimum value  $U$  can attain as a function of the number of players.

### C Personalized offers

The model defined in Section 2 enables each player  $j$  to choose a single item out of  $\mathcal{L}_j$ . In this section, we extend to a more general case, where players may select several items and offer each user the one item which satisfies him the most.

For reader convenience, we repeat the part of the model being reconsidered:

2. The set of items (e.g. possible ad formats/messages to select from) available to player  $j$  is denoted by  $\mathcal{L}_j$ , which we assume to be finite. A *strategy* of player  $j$  is an item from  $\mathcal{L}_j$ .

Consider the case where each player  $j$  is limited to choose up to  $k_j$  items from  $\mathcal{L}_j$ , where  $k_j$  is fixed. Formally, the strategy space of each player  $j$  is  $\{L : |L| \leq k_j, L \subset \mathcal{L}_j\}$ , and we keep on using  $X_j$  to represent her strategy. In addition, users are now targeted personally - for each user  $u_i$ , player  $j$  offers the product with the highest satisfaction level, namely  $\sigma_i(X_j) = \max_{l \in X_j} \sigma_i(l)$ .

We again define the characteristic function of the cooperative game as  $v_i(\mathcal{C}; \mathbf{X}) = \sigma_i(\mathbf{X}_{\mathcal{C}})$ . The coalition payoff is  $u_i$ 's highest satisfaction with items offered by the coalition members. Hence, the Shapley value of each player remains the same, and the proof of Theorem 3 holds as is. In addition, Theorem 5 and Lemma 2 did not make use of this assumption, and hence hold as well.

As for Theorem 2, a few modifications are required. The strategy of selecting a set  $L \in \mathcal{L}$  be modeled as selecting all resources associated with intervals that are subsets of  $[0, \sigma_i(L)]$ , where  $\sigma_i(L) = \max_{l \in L} \sigma_i(l)$ . Namely,

$$\mathcal{A}(L) = \{r_m^i : \sigma_i(L) \geq \epsilon_m, m \in [B], i \in [n]\}.$$

Thus, there is an induced one-to-one function from the power set of items to the power set of resources,  $\mathcal{A} : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{R}}$ . Mapping between items and resources, we define the set of possible strategies of player  $j$ :

$$S_j = \{\mathcal{A}(L) : L : |L| \leq k_j, L \subset \mathcal{L}_j\}.$$

Using these modifications, the proof of Theorem 2 given in Subsection A.3 now holds.